Definitions

Z — set of integers, defined through Axioms 1.1–5. These include the definitions of 0, 1, and $-m$.

m **is divisible by** n (or alternatively, n **divides** m) if there exists $j \in \mathbf{Z}$ such that $m = jn$. We use the notation $n \mid m$.

m is **even** if it is divisible by 2.

$m - n$ is defined to be $m + (-n)$.

N — set of natural numbers (or positive integers), defined through Axiom 2.1.

Let $m, n \in \mathbf{Z}$. The statements $m < n$ (m **is less than** n) and $n > m$ (n **is greater than** m) both mean that

$$n - m \in \mathbf{N}.$$

The notations $m \le n$ (m **is less than or equal to** n) and $n \ge m$ (n **is greater than or equal to** m) mean that

$$m < n \qquad \text{or} \qquad m = n.$$

A is a **subset** of B (in symbols, $A \subseteq B$) means

$$\text{if } x \in A \text{ then } x \in B.$$

Two sets A and B are **equal** (in symbols, $A = B$) if

$$A \subseteq B \qquad \text{and} \qquad B \subseteq A.$$

The **intersection** of two sets A and B is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

The **union** of A and B is

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

For two sets A and B, we define the **set difference**

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

Given a set $A \subseteq X$, we define the **complement** of A in X to be $X - A$. If the bigger set X is clear from the context, one often writes A^c for the complement of A (in X).

The **(Cartesian) product** of the sets A and B is

$$A \times B := \{(a,b) : a \in A \text{ and } b \in B\}.$$

We call (a,b) an **ordered pair**.

$\exists$ — there exist(s)...
$\forall$ — for all...
$\exists!$ — existence and uniqueness

$\heartsuit \Rightarrow \clubsuit$ — if $\heartsuit$ then $\clubsuit$
$\clubsuit \Rightarrow \heartsuit$ is the **converse** of $\heartsuit \Rightarrow \clubsuit$
(not $\clubsuit$) $\Rightarrow$ (not $\heartsuit$) is the **contrapositive** of $\heartsuit \Rightarrow \clubsuit$; this is equivalent to $\heartsuit \Rightarrow \clubsuit$.

Sum. Let $(x_j)_{j=1}^{\infty}$ be a sequence of integers. For each $k \in \mathbf{N}$, we want to define an integer called $\sum_{j=1}^{k} x_j$:

(i) Define $\sum_{j=1}^{1} x_j$ to be x_1.

(ii) Assuming $\sum_{j=1}^{n} x_j$ already defined, we define $\sum_{j=1}^{n+1} x_j$ to be $\left(\sum_{j=1}^{n} x_j\right) + x_{n+1}$.

Product. We define an integer called $\prod_{j=1}^{k} x_j$:

(i) Define $\prod_{j=1}^{1} x_j := x_1$.

(ii) Assuming $\prod_{j=1}^{n} x_j$ defined, we define $\prod_{j=1}^{n+1} x_j := \left(\prod_{j=1}^{n} x_j\right) \cdot x_{n+1}$.

Factorial. We define $k!$ ("k factorial") for all integers $k \geq 0$ by:

(i) Define $0! := 1$.

(ii) Assuming $n!$ defined (where $n \in \mathbf{Z}_{\geq 0}$), define $(n+1)! := (n!) \cdot (n+1)$.

Power. Let b be a fixed integer. We define b^k for all integers $k \geq 0$ by:

(i) $b^0 := 1$.

(ii) Assuming b^n defined, let $b^{n+1} := b^n \cdot b$.

A **relation** on a set A is a subset of $A \times A$. Given a relation $R \subseteq A \times A$, we often write $x \sim y$ instead of $(x,y) \in R$ and we say that x **is related to** y (by the relation R). The relation $R \subseteq A \times A$ is an **equivalence relation** if it has the following three properties:

(i) $a \sim a$ for all $a \in A$. *(reflexivity)*

(ii) $a \sim b$ implies $b \sim a$. *(symmetry)*

(iii) $a \sim b$ and $b \sim c$ imply $a \sim c$. *(transitivity)*

Given an equivalence relation $\sim$ on A, the **equivalence class** of $a \in A$ is

$$[a] := \{b \in A : b \sim a\}.$$