Some Problems for Block 1

We start with some "recreational" math problems that do not require much knowledge. These sorts of problems are fun, and they secretly teach many lessons about mathematics, as you will see. The rest of the problems deal explicitly with our topic, number theory. Your job: mess around and search for patterns and conjectures.

1 You have a fatal disease, but luckily, you have a cure: you need to take one pill A and one pill B together at the same time every day for 10 days. You have a bottle of 10 A pills and a bottle of 10 B pills (and is are no other supply of these pills in the universe). If you take more or less than the proper dose, you will die. For example, if you take more than one A pill at one time, or if you only take a B pill and don't take the A pill, you die.

On the first day, everything goes well. You carefully pour one A pill into your palm and carefully pour one B pill into your palm, and you swallow them together. On the next day, you pour one A pill into your palm, but you aren't careful, and you pour *two* B pills into your palm! Now you have three pills in your palm. And here's the bad news: *the two types of pills are absolutely indistinguishable*. How do you survive?

2 Find the next member in this sequence.

$$1, 11, 21, 1211, 111221, \ldots$$

3 There are 50 cards on a table top. You know that exactly 20 of them are face up, and the rest are face down. How can you separate the cards into two piles, each with the same number of face-up cards, *with your eyes closed*?

4 Lockers in a row are numbered $1, 2, 3, \ldots, 1000$. At first, all the lockers are closed. A person walks by, and opens every other locker, starting with locker #2. Thus lockers $2, 4, 6, \ldots, 998, 1000$ are open. Another person walks by, and changes the "state" (i.e., closes a locker if it is open, opens a locker if it is closed) of every third locker, starting with locker #3. Then another person changes the state of every fourth locker, starting with #4, etc. This process continues until no more lockers can be altered. Which lockers will be closed and which will be open?

5 Here are the first few powers of 2:

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024 \ldots$$

It should be easy to predict the ending digit. For example, what is the ending (rightmost) digit of 2^{2007}? But what about *starting* digits? Are there any patterns? Is there any power of 2 that starts with a 7? How about starting with a 9?

6 Which are there more of, perfect squares or primes? Does this question even make sense?

7 *Twin primes* are pairs of consecutive odd numbers that are prime. For example, 17 and 19 are twin primes. Also, 29 and 31 are twin primes. Is there an analogous notion of "triplet primes?" Why or why not?

8 The year 2002 was pretty cool. The next year that shares the same property that made 2002 cool is 2010. The only other years with the same property during this century are 2030, 2046, and 2090. What do these years all have in common?

9 Here are the first few rows of **Pascal's Triangle**.

$$\begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&2&&1&&& \\ &&1&&3&&3&&1&& \\ &1&&4&&6&&4&&1& \\ 1&&5&&10&&10&&5&&1, \end{array}$$

where the elements of each row are the sums of pairs of adjacent elements of the prior row. For example, $10 = 4+6$. The next row in the triangle will be

$$1, 6, 15, 20, 15, 6, 1.$$

There are many interesting patterns in Pascal's Triangle. Discover as many patterns and relationships as you can, and prove as much as possible. In particular, can you somehow extract the Fibonacci numbers (see next problem) from Pascal's Triangle (or vice versa)? Another question: is there a pattern or rule for the **parity** (evenness or oddness) of the elements of Pascal's Triangle?

10 The **Fibonacci numbers** f_n are defined by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. For example, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, $f_5 = 5$, $f_6 = 8$, $f_7 = 13$, $f_8 = 21$. Play around with this sequence; try to discover as many patterns as you can, and try to prove your conjectures as best as you can. As with the previous problem, think about parity. Parity just involves divisibility by 2. What about other divisibilities (for example, divisibility by 5)?.