Math Camp – Some Number Theory Problems

July 2008

(1) There are 1000 lockers in a row, numbered 1, 2, . . . , 1000. At first, all the lockers are closed. A person walks by and opens every other locker, starting with locker #2. Another person walks by and changes the "state" (i.e., closes a locker that is open and opens a locker that is closed) of every third locker, starting with locker #3. Then another person changes the state of every 4th locker, starting with locker #4, etc. This process continues until no more lockers can be altered. Which lockers will be closed?

(2) There are 50 cards on a table. You know that exactly 20 of them face up and the rest are face down. How can you separate the cards into two piles, each with the same number of face-up cards, with your eyes closed?

(3) Which are there more of, perfect squares or primes? Does this question even make sense?

(4) *Twin primes* are pairs of consecutive odd numbers that are prime. For example, 17 and 19 are twin primes, as are 29 and 31. Is there an analogous notion of *triplet primes*? Why or why not?

(5) Find the remainder when 2008^{2008} is divided by

(a) 2
(b) 3
(c) 5
(d) 7
(e) 11
(f) 13

(6) Since $24 = 3 + 5 + 7 + 9$, the number 24 can be written as the sum of at least two consecutive odd positive integers.

(a) Can 2008 be written as the sum of at least two consecutive odd positive integers? If so, give an example of how it can be done, if not, provide a proof why not.
(b) Can 2009 be written as the sum of at least two consecutive odd positive integers? If so, give an example of how it can be done, if not, provide a proof why not.

(7) An integer is a *perfect number* if it is equal to the sum of all of its proper divisors. For example, 28 is perfect because $28 = 1 + 2 + 4 + 7 + 14$. Find all perfect numbers that are also factorials.

(8) Prove that if p is a prime number then $(p - 1)! + 1$ will be divisible by p. For example, if $p = 7$, then

$$(p - 1)! + 1 = 721 = 7 \times 103\,.$$

(9) In Klopstockia, money only comes in two denominations, \$5 and \$7.

(a) Find two ways to pay \$100 without getting change.
(b) Can you find *all* solutions to (a) in a systematic way?