MATH 325 Final Exam 12/18/24 12:30–2:30 p.m.

Name: ______________________________

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible.

1. Let

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 3 \\ 2 & 3 & 7 & 0 \end{bmatrix}.$$

   (a) Compute a basis for the kernel of A.

   (b) Compute a basis for the image of A, i.e., the span of its columns.

   (c) Is the vector $\begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}$ in the image of A? Justify your answer.

(a) $$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & 3 & 0 \\ 2 & 3 & 7 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \operatorname{rank}(A) = 2$ and $\dim \ker(A) = 4-2=2$

$$\ker(A) = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 3 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(b) $\dim \operatorname{im}(A) = 4 - \dim \ker(A) = 2$

$$\operatorname{im}(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}$$

(c) $$\begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \in \operatorname{im}(A)$$

2. (a) Give three examples of vector spaces, different from $\mathbb{R}^n$ for some n. For each of them, describe the zero vector and the additive inverse of an arbitrary vector.

(b) Which ones of the following subsets of $\mathbb{R}^{2\times 2}$ are subspaces? Justify your answers.

(i) the set of 2×2 diagonal matrices;

(ii) the set of 2×2 singular matrices (i.e., not of full rank);

(iii) the set of 2×2 upper-triangular matrices.

(a) • polynomials with coefficients in $\mathbb{R}$

0 is the zero polynomial, $-p(x)$ has all coefficient negated

• $m \times n$ matrices over $\mathbb{R}$

0 is the zero matrix, $-A$ has all entries negated

• functions $\mathbb{R} \to \mathbb{R}$

0 is the zero function, $-f(x)$ is the negation

(b) (i) Yes: if A and B are diagonal matrices, so is $\lambda A + B$ for $\lambda \in \mathbb{R}$

(ii) No: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ do not have full rank, but $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ does

(iii) Yes: if A and B are upper Δ, so is $\lambda A + B$ for $\lambda \in \mathbb{R}$.

3. Let W be the subspace of $\mathbb{R}^3$ spanned by $\overset{v_1}{\begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}}$ and $\overset{v_2}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$.

(a) Compute an orthogonal basis of W.

(b) Find the vector $x \in \mathbb{R}^2$ that minimizes

$$\left\| \begin{bmatrix} -3 & 1 \\ 4 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\|$$

where the norm comes from the standard dot product.

(a) Gram-Schmidt: $u_1 = v_1 = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$

$[\|u_1\| = 5]$

$$u_2 = v_2 - \frac{\langle u_1, v_2 \rangle}{\|u_1\|^2} u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{25} \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 28/25 \\ 21/25 \\ 1 \end{bmatrix} \qquad \left[\|u_2\| = \tfrac{1}{5}\sqrt{74}\right]$$

(b) Project $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ onto W.

$$w = \underbrace{\left\langle \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{5}\begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} \right\rangle}_{\frac{1}{5}\cdot 5 = 1} \cdot \frac{1}{5}\begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$$

$$+ \underbrace{\left\langle \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \frac{5}{\sqrt{74}}\begin{bmatrix} 28/25 \\ 21/25 \\ 1 \end{bmatrix} \right\rangle}_{\frac{5}{\sqrt{74}} \cdot \frac{19}{5} = \frac{19}{\sqrt{74}}} \cdot \frac{5}{\sqrt{74}}\begin{bmatrix} 28/25 \\ 21/25 \\ 1 \end{bmatrix}$$

$$= \frac{1}{74}\begin{bmatrix} 62 \\ 139 \\ 95 \end{bmatrix}$$

4. Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

(a) Compute the eigenvalues and eigenspaces of A.

(b) Is A diagonizable? If so, compute Q and a diagonal matrix D such that $A = QDQ^{-1}$.

(a) $$\det\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix} = (1-\lambda)\left((2-\lambda)(1-\lambda)-1\right) + (-1)(1-\lambda)$$

$$= (1-\lambda)\left(2-3\lambda+\lambda^2-2\right)$$

$$= (1-\lambda)\lambda(\lambda-3)$$

$\rightarrow$ eigenvalues 0, 1, 3

$\rightarrow$ each comes with a 1-dimensional eigenspace

$\lambda_1 = 0$: $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$ $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \rightarrow v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

eigenspaces:
span$\{v_1\}$
span$\{v_2\}$
span$\{v_3\}$

$\lambda_3 = 3$: $\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} -2 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

(b) Yes. $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\leftarrow$ normalized eigenvectors in columns

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -2/\sqrt{6} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$