MATH 325 Midterm Exam 10/16/24 12:00–1:40 p.m.

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible.

1. Consider the following system of equations.

$$\begin{array}{rcrcrcrcrcr} 2x_1 & - & 3x_2 & - & 7x_3 & + & 5x_4 & + & 2x_5 & = & -2 \\ x_1 & - & 2x_2 & - & 4x_3 & + & 3x_4 & + & x_5 & = & -2 \\ 2x_1 & & & - & 4x_3 & + & 2x_4 & + & x_5 & = & 3 \end{array}$$

   (a) Compute the general solution to this system. Make sure to identify basic and free variables.

   (b) Write down one specific solution to this system.

   (c) Let

$$A = \begin{bmatrix} 2 & -3 & -7 & 5 & 2 \\ 1 & -2 & -4 & 3 & 1 \\ 2 & 0 & -4 & 2 & 1 \end{bmatrix}.$$

   What is the rank of A and what is the dimension of the kernel of A? Explain.

(a) Gauss : $\left[\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ 0 & -1 & -1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \rightarrow$

$x_5 = -1$

$x_2 = 2 - x_3 + x_4$

$x_1 = -1 + \frac{3}{2}(2 - x_3 + x_4) + 7x_3 - 5x_4$

$= 2 + 5\frac{1}{2}x_3 - 3\frac{1}{2}x_4$

basic : x_1, x_2, x_5

free : x_3, x_4

(b) $\begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

(c) $\text{rank}(A) = \#\text{ pivots} = 3$

$\dim \ker(A) = 5 - \text{rank}(A) = 2$

2. Let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 5 \\ 2 & 3 & 2 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix}$$

(a) Is B invertible? Is C invertible? Explain.

(b) Compute the inverse(s) of B and of C, if they exist.

(c) Find a basis for the kernel of B.

(a+b) B has rank 2 (the middle row is the sum of the other two rows) and so is not invertible.

C:
$$\left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 4 & 2 & 3 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & -3 & 2 & 0 \\ 0 & -1 & 0 & -2 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -5 & 3 & 1 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\rightarrow C^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ 2 & -1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

(c) $\operatorname{rank}(B) = 2 \rightarrow \dim \ker(B) = 1$, so we need to find one solution to $Bx = 0$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 5 & 5 & 0 \\ 2 & 3 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -1 & -4 & 0 \end{array}\right]$$

$$x_2 = -4x_3$$
$$x_1 = (-2)(-4x_3) - 3x_3 = 5x_3$$

$$\ker(B) = \operatorname{span}\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

3. Let $P^{(2)}$ be the vector space of all polynomials over $\mathbb{R}$ of degree ≤ 2.

(a) What is the dimension of $P^{(2)}$? (No justification necessary.)

(b) Determine whether $x-1$, $-x^2+1$, and $-3x^2+3x$ are linearly independent. Justify your answer.

(c) Determine whether $x-1$, $-x^2+1$, and $-3x^2+3x$ span $P^{(2)}$. Justify your answer.

(a) 3

(b) $\lambda_1(x-1) + \lambda_2(-x^2+1) + \lambda_3(-3x^2+3x) = 0$

$$\underbrace{(-\lambda_2 - 3\lambda_3)}_{=0} x^2 + \underbrace{(\lambda_1 + 3\lambda_3)}_{=0} x + \underbrace{(-\lambda_1 + \lambda_2)}_{=0} = 0$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\lambda_2 = -3\lambda_3 \qquad \lambda_1 = -3\lambda_3 \qquad \lambda_1 = \lambda_2$$

$\rightarrow$ we can find nonzero λ, e.g. $\lambda = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$

$\rightarrow$ the three polynomials are lin. dependent

(c) Because the 3 polynomials are lin. dependent, they cannot span $P^{(2)}$.

4. (a) Give a basis for the vector space of all symmetric 3×3 matrices and deduce its dimension.

(b) Let A be an $m \times n$ matrix. Show that AA^T is symmetric. (*Hint:* use the rules of transposition.)

(c) True or false: if A and B are both symmetric $n \times n$ matrices, then AB is also symmetric. Either give a proof or give a counterexample.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \text{dim } 6$

(b) $(AA^T)^T = (A^T)^T A^T = AA^T \rightarrow AA^T$ is symmetric

(c) false: one counterexample is

$$\begin{array}{ccc} A & B & AB \\ \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = & \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \end{array}$$