MATH 420/720 Final Exam 5/19/25 12:30–2:30 p.m.

Name: ______________________________

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. Part (a) of each question is worth 5 points, part (b) is worth 10 points.

If you are in Math 420, I will drop the lowest total score among the four problems.

(1) (a) Show that there are exactly $\binom{n-1}{k-1}$ compositions (i.e., ordered partitions) of n with k parts.

(b) Compute a formula for the number of compositions of n with k parts, all of which are ≥ 2. (*Hint:* there is more than one way to construct such a formula, including bijectively and via inclusion–exclusion.)

Solution:

(a) The number of compositions of n with k parts is

$$\#\left\{x \in \mathbb{Z}_{>0}^k : x_1 + x_2 + \cdots + x_k = n\right\} = \#\left\{x \in \mathbb{Z}_{\geq 0}^k : x_1 + x_2 + \cdots + x_k = n - k\right\}$$
$$= \binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}.$$

(b) **Bijective approach.** We are looking for

$$\#\left\{x \in \mathbb{Z}_{\geq 2}^k : x_1 + x_2 + \cdots + x_k = n\right\} = \#\left\{x \in \mathbb{Z}_{\geq 0}^k : x_1 + x_2 + \cdots + x_k = n - 2k\right\}$$
$$= \binom{n-2k+k-1}{k-1} = \binom{n-k-1}{k-1}.$$

Inclusion–exclusion approach. Let S be the set of all compositions of n with k parts, and let S_j consist of all such compositions whose j'th part is 1. Then S_I consists of compositions with $|I|$ parts equal to 1, and so $|S_I| = \binom{n-|I|-1}{k-|I|-1}$. By the Principle of Inclusion–Exclusion, the number we are after is

$$\left|S - \bigcup_{j=1}^{k} S_j\right| = \sum_{I \subseteq [k]} (-1)^{|I|} |S_I| = \sum_{I \subseteq [k]} (-1)^{|I|} \binom{n-|I|-1}{k-|I|-1}$$
$$= \sum_{j=0}^{k-1} (-1)^j \binom{n}{j} \binom{n-j-1}{k-j-1}$$
$$= \sum_{j=0}^{k-1} (-1)^j \frac{n!(n-j-1)!}{j!(n-j)!(k-j-1)!(n-k)!}$$
$$= \frac{n!}{(n-k)!} \sum_{j=0}^{k-1} \frac{(-1)^j}{j!(k-j-1)!(n-j)}.$$

(Yes, the two formulas are different; so they yield an identity...)

(2) (a) Recall that a *Dyck path* of length n is a northeast lattice path from $(0,0)$ to (n,n) that never goes below the diagonal $y=x$. Compute a recurrence relation for the number $c(n)$ of Dyck paths of length n.

(b) Compute a closed form of the generating function of $c(n)$.

Solution:

(a) The initial condition is $c(0)=1$, and the recurrence

$$c(n) \;=\; \sum_{j=1}^{n} c(j-1)\,c(n-j)$$

stems from going through the n possibilities that the path touches the diagonal $y=x$ first in the point (j,j). In this case, the number of paths from $(0,0)$ to (j,j) is $c(j-1)$ because this part needs to be *strictly* above the diagonal except at the end points; the number of paths from (j,j) to (n,n) is $c(n-j)$.

(b) Rewriting the recurrence as

$$c(n+1) \;=\; \sum_{j=0}^{n} c(j)\,c(n-j)$$

yields the generating function equation

$$\frac{1}{x}\,(C(x)-1) \;=\; \sum_{n\geq 0} c(n+1)\,x^n \;=\; \sum_{n\geq 0}\sum_{j=0}^{n} c(j)\,c(n-j)\,x^n \;=\; C(x)^2.$$

This gives $x\,C(x)^2 - C(x) + 1 = 0$ and thus

$$C(x) \;=\; \frac{1-\sqrt{1-4x}}{2x}$$

(the sign in front of the square root stems from $C(0)=1$).

(3) (a) Compute the exponential generating function for the sequence $a_n = n$.

(b) The *Bernoulli polynomials* $B_k(n)$ are defined through the exponential generating function

$$\frac{x\,e^{nx}}{e^x-1} \;=\; \sum_{k\geq 0} B_k(n)\,\frac{x^k}{k!}\,.$$

The *Bernoulli numbers* are $B_k := B_k(0)$ (and thus come with the exponential generating function $\frac{x}{e^x-1}$). Prove that

$$\sum_{j=0}^{n-1} j^{k-1} \;=\; \tfrac{1}{k}\left(B_k(n)-B_k\right).$$

(*Hint:* start by expressing $\sum_{j=0}^{n-1} e^{jx}$ in two ways: by expanding e^{jx} into an exponential series, and as a finite geometric series in the variable e^x.)

Solution:

(a)

$$\sum_{n\geq 0} n\,\frac{x^n}{n!} \;=\; \sum_{n\geq 1}\frac{x^n}{(n-1)!} \;=\; x\,e^x .$$

(b) On the one hand,

$$\sum_{j=0}^{n-1} e^{jx} \;=\; \sum_{j=0}^{n-1}\sum_{k\geq 0}\frac{(jx)^k}{k!} \;=\; \sum_{k\geq 0}\left(\sum_{j=0}^{n-1} j^k\right)\frac{x^k}{k!}\,.$$

On the other hand,

$$\sum_{j=0}^{n-1} e^{jx} \;=\; \frac{e^{nx}-1}{e^x-1} \;=\; \frac{1}{x}\sum_{k\geq 0}\left(B_k(n)-B_k\right)\frac{x^k}{k!} \;=\; \sum_{k\geq 0}\frac{1}{k}\left(B_k(n)-B_k\right)\frac{x^{k-1}}{(k-1)!}\,.$$

Matching the coefficients in both expansions yields the identity.

(4) (a) Define the Möbius function $\mu(x,z)$ for the poset $(P,\preceq)$.

(b) Now let $P = B_n$, consisting of all subsets of $[n]$, with the relation $\subseteq$. Show that $\mu(S,T) = (-1)^{|T\setminus S|}$.

Solution:

(a) For $x \preceq z$, the Möbius function is defined recursively via

$$\sum_{x\preceq y\preceq z}\mu(x,y) \;=\; \begin{cases} 1 & \text{if } x = z,\\ 0 & \text{otherwise.}\end{cases}$$

(b) For starters, $(-1)^{|T\setminus S|} = 1$ when $S = T$. For $S \subsetneq T$,

$$\sum_{S\subseteq X\subseteq T}(-1)^{|X\setminus S|} \;=\; \sum_{\varnothing\subseteq X\subseteq T\setminus S}(-1)^{|X|} \;=\; \sum_{k=0}^{m}(-1)^k\binom{m}{k} \;=\; (1-1)^m \;=\; 0$$

where $m := |T\setminus S|$. Thus $(-1)^{|T\setminus S|}$ satisfies all requirements of the Möbius function.