MATH 420/720 Generating Function Exercises

(1) Given a sequence $(a_n)_{n\geq 0}$ with generating function $A(x) := \sum_{n\geq 0} a_n\, x^n$, let

$$B(x) = \sum_{n\geq 0} b_n\, x^n = \frac{A(x)}{1-x}\,.$$

Find a formula for b_n.

(2) Use the previous exercise to prove that the Fibonacci numbers satisfy

$$f_0 + f_1 + \cdots + f_n = f_{n+2} - 1\,.$$

(3) Let $c(n)$ denote the number of compositions of n.

(a) Show that $c(n) = 2^{n-1}$.

(b) Recall that in class we derived the generating function for $c_A(n)$, the number of compositions of n with parts in a given set A. Confirm that for $A = \mathbb{Z}_{>0}$ this generating function gives rise to your formula in part (a).

(4) Now let $A = \{1, 2, 4, 5, 7, 8, \dots\}$, the set of all positive integers that are not multiples of 3.

(a) Compute the generating function for $c_A(n)$, the number of compositions of n with parts in A.

(b) Derive a recurrence relation for $c_A(n)$ and argue that these numbers should be called *Tribonacci numbers*.

(5) Find the sum of the first n squares by differentiating the generating function $\displaystyle\sum_{k=0}^{n} x^k$.