MATH 420/720 Binomial Coefficients Make a Comeback

(1) Recall that the Eulerian numbers $\operatorname{eul}(k,j)$ are defined via $\operatorname{eul}(0,0)=1$ and

$$\sum_{n\ge 1} n^k\, x^n \;=\; \frac{x\sum_{j=0}^{k-1}\operatorname{eul}(k,j)\,x^j}{(1-x)^{k+1}}\,.$$

Prove that

(a) $\operatorname{eul}(k,j) \;=\; \sum_{m=0}^{j}(-1)^m\binom{k+1}{m}(j+1-m)^k\,;$

(b) $n^k \;=\; \sum_{m=0}^{k-1}\operatorname{eul}(k,m)\binom{n+m}{k}\,.$

(*Hint:* for the first identity, expand $(1-x)^{k+1}\sum_{n\ge1} n^k\,x^{n-1}$, for the second, expand $(1-x)^{-k-1}\sum_{n\ge1} n^k\,x^{n-1}$. There are also nongeneratingfunction ways to prove these...)

(2) Recall that $\left[{n \atop k}\right]$ is the Stirling number of the first kind. Show that

$$\binom{m}{n} \;=\; \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\left[{n \atop k}\right] m^k.$$

(3) Recall that $\left\{{n \atop k}\right\}$ is the Stirling number of the second kind. Show that

$$n^k \;=\; \sum_{j=0}^{k}\left\{{k \atop j}\right\} j!\binom{n}{j}\,.$$

(4) (a) Suppose n and k are integers such that $0\le k\le n$. Assume you didn't know anything about the binomial coefficient $\binom{n}{k}$, except the relations

$$\binom{n}{k} \;=\; \binom{n-1}{k-1}+\binom{n-1}{k} \qquad\text{and}\qquad \binom{n}{0}=1\,.$$

Compute the generating function

$$B_n(x) \;:=\; \sum_{k\ge0}\binom{n}{k}x^k$$

and use it to find a formula for $\binom{n}{k}$.

(b) Convince yourself that your computation is identical if we allow n to be any complex number and k to be any nonnegative integer.

(c) Compute the generating function

$$B(x) \;:=\; \sum_{n\ge0}\sum_{k\ge0}\binom{n}{k}x^k y^n$$

and use it to find a formula for the generating function $\beta_k(y) := \sum_{n\ge k}\binom{n}{k}y^n$.