MATH 420/720 **Two Exponential Generating Functions & Geometry**

(1) Recall once more that $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind. Show that

$$\sum_{n\ge 0} \left\{ {n \atop k} \right\} \frac{x^n}{n!} = \frac{1}{k!} \left(e^x - 1\right)^k .$$

(2) The *Bernoulli numbers* B_k are defined through the generating function

$$\frac{z}{e^z - 1} = \sum_{k\ge 0} B_k \frac{z^k}{k!} .$$

(a) Compute B_0, B_1, and B_2.

(b) Prove that $B_d = 0$ for all odd $d \ge 3$. (*Hint:* show that $\frac{z}{e^z-1} + \frac{1}{2} z$ is an even function.)

(3) The *Bernoulli polynomials* $B_k(x)$ are defined through the generating function

$$\frac{z\, e^{xz}}{e^z - 1} = \sum_{k\ge 0} B_k(x) \frac{z^k}{k!} .$$

(Thus the Bernoulli numbers are $B_k = B_k(0)$.)

(a) Show that for integers $k \ge 1$ and $n \ge 2$,

$$\sum_{j=0}^{n-1} j^{k-1} = \frac{1}{k} \left(B_k(n) - B_k\right) .$$

(*Hint:* play with the exponential generating function of $B_k(n) - B_k$.)

(b) Show that for each positive integer n,

$$n\, x^{n-1} = \sum_{k=1}^{n} \binom{n}{k} B_{n-k}(x) .$$

(c) Complementarily, show that

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k\, x^{n-k}.$$

(4) Let $\mathcal{R}$ be an lattice rectangle whose edges are parallel to the coordinate axes, and let $\mathcal{T}$ be a rectangular lattice triangle two of whose edges are parallel to the coordinate axes. Show that Pick's theorem holds for $\mathcal{R}$ and $\mathcal{T}$.