MATH 420/720 Exam 1 2/26/25 1:00–1:50 p.m.

Name: ______________________________

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. Part (a) of each question is worth 5 points, as are parts (1b) and (1c). Each other part (b) is worth 10 points.

If you are in Math 420, pick three of the following four problems. (If you end up working on all four problems, please mark clearly which one should not be considered.)

(1) (a) Define the Stirling number of the second kind $S(n,k)$ via a combinatorial instance counted by $S(n,k)$.

(b) You have 10 M&M's, all of different colors. Give a formula for the number of ways of dividing your M&M's into various piles. (You don't need to compute the ingredients of your formula.)

(c) Repeat the last problem for the case that all of your 10 M&M's have the same color. (You again don't need to compute the ingredients of your formula.)

Solution:

(a) $S(n,k)$ counts the number of ways of partitioning n ordered objects into k unordered parts.

(b) $\sum_{k=1}^{10} S(10,k)$.

(c) This is equivalent to counting all integer partitions of 10, i.e., $p(10,10)$.

(2) (a) Define $\binom{n}{k}$ via a combinatorial instance counted by $\binom{n}{k}$.

(b) Let E denote the collection of subsets of $[n]$ with even cardinality, and let O consists of the subsets of $[n]$ with odd cardinality. Define $f : E \to O$ via

$$f(S) \;=\; \begin{cases} S \cup \{n\} & \text{if } n \notin S, \\ S \setminus \{n\} & \text{if } n \in S. \end{cases}$$

Show that f is a bijection and deduce that $\displaystyle\sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k}$.

($\star$) *Bonus question:* can you think of a simpler proof?

Solution:

(a) $\binom{n}{k}$ counts the number of k-subsets of $[n]$.

(b) The function $g : O \to E$ defined by

$$g(S) \;=\; \begin{cases} S \cup \{n\} & \text{if } n \notin S, \\ S \setminus \{n\} & \text{if } n \in S \end{cases}$$

is the inverse of f. Thus f is a bijection and consequently

$$|E| \;=\; \sum_{k \text{ even}} \binom{n}{k} \;=\; |O| \;=\; \sum_{k \text{ odd}} \binom{n}{k} .$$

($\star$) The identity also follows from $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = (1-1)^n = 0$.

(3) (a) Define $\left(\!\binom{n}{k}\!\right)$ via a combinatorial instance counted by $\left(\!\binom{n}{k}\!\right)$.

(b) Show that $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$.

Solution:

(a) $\left(\!\binom{n}{k}\!\right)$ counts the k-multisubsets of $[n]$.

(b) Ordering the k element in a given multisubset gives

$$\begin{aligned}\left(\!\binom{n}{k}\!\right) &= \left|\left\{(x_1, x_2, \dots, x_k) \in \mathbb{Z}^k : 1 \le x_1 \le x_2 \le \cdots \le x_k \le n\right\}\right| \\ &= \left|\left\{(y_1, y_2, \dots, y_k) \in \mathbb{Z}^k : 1 \le y_1 < y_2 < \cdots < y_k \le n+k-1\right\}\right| \\ &= \binom{n+k-1}{k}\end{aligned}$$

via the bijection $y_j := x_j + j - 1$.

(4) (a) Define what it means for a graph $G = (V, E)$ to be a tree.

(b) Define the *average degree* of a graph $G = (V, E)$ as

$$a(G) \ := \ \frac{1}{|V|}\sum_{v \in V} \deg(v)\,.$$

Find (and prove) an expression for the number $|V|$ of vertices of a tree T in terms of $a(T)$.

Solution:

(a) A tree is a connected graph without cycles.

(b) Since in a tree we have $|V| = |E| + 1$,

$$a(T) \ = \ \frac{1}{|V|}\sum_{v \in V} \deg(v) \ = \ \frac{1}{|V|} \cdot 2|E| \ = \ \frac{2(|V|-1)}{|V|}$$

and so $|V| = \dfrac{2}{2 - a(T)}$.