MATH 420/720 Exam 2 4/4/25 1:00–1:50 p.m.

Name: ______________________________

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. Part (a) of each question is worth 5 points, part (b) is worth 10 points.

If you are in Math 420, pick three of the following four problems. (If you end up working on all four problems, please mark clearly which one should not be considered.)

(1) (a) State the Principle of Inclusion–Exclusion.

(b) Let $D(n)$ be the number of permutations of $[n]$ without a fixed point. Compute $D(n)$.

Solution:

(a) Given a finite set S with subsets $S_1, S_2, \dots, S_n$,

$$\left| S - \bigcup_{j=1}^{n} S_j \right| = |S| - \sum_{j \in [n]} |S_j| + \sum_{j \neq k \in [n]} |S_j \cap S_k| - \cdots \pm |S_1 \cap S_2 \cap \cdots \cap S_n| \, .$$

(b) Let S_j consist of all permutations of $[n]$ that fix j. By the Principle of Inclusion–Exclusion,

$$\begin{aligned} D(n) &= n! - \sum_{j \in [n]} |S_j| + \sum_{j \neq k \in [n]} |S_j \cap S_k| - \cdots \pm |S_1 \cap S_2 \cap \cdots \cap S_n| \\ &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots \pm 1 \\ &= n! \left(1 - 1 + \frac{1}{2} - \frac{1}{3!} + \cdots \pm \frac{1}{n!} \right) . \end{aligned}$$

(2) (a) Define what it means for a sequence (a_n) to be log concave.

(b) Fix an integer $k \geq 0$. Prove that $\binom{n}{k}$, viewed as a sequence in n, is log concave.

Solution:

(a) $a_n^2 \geq a_{n-1} a_{n+1}$.

(b)

$$\begin{aligned} \frac{\binom{n}{k}^2}{\binom{n+1}{k}\binom{n-1}{k}} &= \frac{n!^2 k!^2 (n+1-k)!\,(n-1-k)!}{k!^2 (n-k)!^2 (n+1)!\,(n-1)!} = \frac{n(n+1-k)}{(n-k)(n+1)} \\ &= \frac{n^2 - nk + n}{n^2 - nk + n - k} \geq 1 \, . \end{aligned}$$

(3) (a) State the condition under which a given generating function $A(x) := \sum_{n \geq 0} a_n \, x^n$ has a multiplicative inverse, and give an idea how to compute it.

(b) Let $(a_n)_{n\geq 0}$ be recursively defined through

$$a_0 = 0 \qquad \text{and for } n \geq 1, \qquad a_n = 2\,a_{n-1} + 3\,.$$

Compute the generating function $A(x) := \sum_{n\geq 0} a_n\, x^n$ and deduce a closed form for a_n.

You may use (without confirming it) the partial-fraction decomposition

$$A(x) \;=\; \frac{3x}{(1-x)(1-2x)} \;=\; \frac{-3}{1-x} + \frac{3}{1-2x}\,.$$

Solution:

(a) The condition is $a_0 \neq 0$. Writing the inverse of $A(x)$ as $B(x) := \sum_{n\geq 0} b_n\, x^n$ gives the condition

$$a_0\, b_0 \;=\; 1$$

from which we can compute b_0, then

$$a_0\, b_1 + a_1\, b_0 \;=\; 0$$

from which we can compute b_1, etc.

(b) By the given conditions for (a_n),

$$A(x) \;=\; \sum_{n\geq 1} a_n\, x^n \;=\; 2\sum_{n\geq 1} a_{n-1}\, x^n + 3\sum_{n\geq 1} x^n \;=\; 2x\, A(x) + 3\,\frac{x}{1-x}$$

and so

$$A(x) \;=\; \frac{3x}{(1-x)(1-2x)} \;=\; \frac{-3}{1-x} + \frac{3}{1-2x} \;=\; -3\sum_{n\geq 0} x^n + 3\sum_{n\geq 0} (2x)^n$$

yielding $a_n = 3(2^n - 1)$.

(4) (a) Define the geometric series and give the generating function for the number of integer partitions using only the part 34.

(b) Fix n. Compute the generating function for the number of k-multisubsets on $[n]$, and deduce that

$$\sum_{k\geq 0} \left(\!\binom{n}{k}\!\right) x^k \;=\; \frac{1}{(1-x)^n}\,.$$

Solution:

(a) The geometric series is

$$\sum_{k\geq 0} x^k \;=\; \frac{1}{1-x}$$

and (thus) the generating function for the number of integer partitions using only the part 34 equals $\frac{1}{1-x^{34}}$.

(b) A given multisubset of $[n]$ has a certain number of 1s, a certain number of 2s, etc., up to a certain number of ns. Thus

$$\sum_{k\geq 0} \left(\!\binom{n}{k}\!\right) x^k \;=\; \left(1 + x + x^2 + \cdots\right)^n \;=\; \frac{1}{(1-x)^n}\,.$$