MATH 420/720 Exam 3 4/30/25 1:00–1:50 p.m.

Name: ______________________________

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. Part (a) of each question is worth 5 points, part (b) is worth 10 points.

If you are in Math 420, I will drop the lowest total score among the three problems.

(1) (a) Given the exponential generating function $A(x) := \sum_{n \ge 0} a_n \frac{x^n}{n!}$, compute $A'(x)$.

(b) Let $(a_n)_{n \ge 0}$ be recursively defined through

$$a_0 = 0 \qquad \text{and for } n \ge 1, \qquad a_n = 2\,a_{n-1} + 3\,.$$

Compute the exponential generating function of a_n and deduce a closed form for a_n. (If you run into a differential equation, try the function $A(x) = b\,e^{2x} + c\,e^x$ and then determine b and c.)

Solution:

(a) We differentiate:

$$A'(x) \;=\; \sum_{n \ge 1} a_n \frac{n\,x^{n-1}}{n!} \;=\; \sum_{n \ge 0} a_{n+1} \frac{x^n}{n!}\,.$$

(b) The given condition $a_{n+1} = 2a_n + 3$ translates in exponential-generating-function-land to

$$A'(x) \;=\; 2\,A(x) + 3e^x$$

and so the *ansatz* $A(x) = b\,e^{2x} + c\,e^x$ gives

$$2b\,e^{2x} + c\,e^x \;=\; 2\left(b\,e^{2x} + c\,e^x\right) + 3e^x$$

yielding $c = -3$ and (because $A(0) = a_0 = 0$) $b = 3$. Thus

$$A(x) \;=\; 3e^{2x} - 3e^x \;=\; 3\sum_{n \ge 0} \frac{(2x)^n}{n!} - 3\sum_{n \ge 0} \frac{x^n}{n!} \;=\; \sum_{n \ge 0} 3\left(2^n - 1\right) \frac{x^n}{n!}$$

and so $a_n = 3(2^n - 1)$.

(2) (a) Recall that an *involution* is a permutation in S_n that factors into 2-cycles. Let i_n be the number of involutions in S_n. Show that $i_0 = i_1 = 1$ and for $n \ge 2$

$$i_n \;=\; i_{n-1} + (n-1)\,i_{n-2}\,.$$

(b) Compute the exponential generating function of i_n.

Solution:

(a) An involution in S_n either fixes n or has a 2-cycle (jn) for some $1 \le j \le n-1$. Thus $i_n = i_{n-1} + (n-1)\, i_{n-2}$.

(b) Let $I(x) := \sum_{n\ge 0} i_n \frac{x^n}{n!}$. The recursion in (a) yields

$$\sum_{n\ge1} i_{n+1} \frac{x^n}{n!} = \sum_{n\ge1} i_n \frac{x^n}{n!} + \sum_{n\ge1} n\, i_{n-1} \frac{x^n}{n!} = \sum_{n\ge1} i_n \frac{x^n}{n!} + \sum_{n\ge0} i_n \frac{x^{n+1}}{n!},$$

that is, (with $i_0 = i_1 = 1$)

$$I'(x) - 1 = I(x) - 1 + x\, I(x),$$

which gives

$$\frac{I'(x)}{I(x)} = 1 + x \qquad \Longrightarrow \qquad \log I(x) = x + \frac{1}{2} x^2.$$

(There is no extra constant, because $\log I(0) = 0$.) Thus $I(x) = e^{x + \frac{1}{2}x^2}$.

(3) (a) Define what it means for the posets $(P, \preceq_P)$ and $(Q, \preceq_Q)$ to be isomorphic.

(b) Consider the divisor lattice D_n of all divisors of a given integer n. Show that the interval $[d, e]$ in D_n is isomorphic to $D_{\frac{e}{d}}$.

Solution:

(a) There is a bijection $f : P \to Q$ such that both f and f^{-1} are order preserving.

(b) Suppose $d|e|n$. Then any number in $[d, e]$ is divisible by d and in turn divides e. Thus the map $f : [d, e] \to D_{\frac{e}{d}}$ defined via

$$f(k) := \frac{k}{d}$$

is well defined. Even better, it is a bijection (with inverse $f^{-1}(k) = dk$), and both f and f^{-1} are visibly order preserving.