MATH 430 **Final Exam** **19 December 2018** **8:00–10:15 a.m.**

Name: ______________________________

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. You may refer to theorems in the book and class notes.

1. Consider the linear optimization problem

$$\begin{aligned} \min \quad & [3,2,1,2,3]\,\mathbf{x} \\ \text{subject to} \quad & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 7 \end{bmatrix} \\ & \mathbf{x} \geq 0\,. \end{aligned}$$

(a) Compute an optimal solution and the optimal cost.

(b) State the dual problem, and confirm that $[-2, 1]$ is an optimal dual solution.

(c) Now replace the vector $\mathbf{b} = \left[\begin{smallmatrix} 2 \\ 7 \end{smallmatrix}\right]$ in the primal problem by a generic vector $\mathbf{b} = \left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right]$. Give a criterion for the primal problem that guarantees that the corresponding dual problem is bounded.

(d) *Extra credit:* Compute conditions for b_1 and b_2 that guarantee that the dual problem is bounded.

Solution.

(a) Two optimal solutions are $[0, 0, 1, 1, 0]$ and $[0, 0, \frac{3}{2}, 0, \frac{1}{2}]$, with optimal cost 3.

(b) The dual problem is

$$\begin{aligned} \max \quad & \mathbf{p} \begin{bmatrix} 2 \\ 7 \end{bmatrix} \\ \text{subject to} \quad & \mathbf{p} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \leq [3,2,1,2,3]\,. \end{aligned}$$

We can see that $[-2, 1]$ satisfies these constraints, and its corresponding cost is 3, so by the strong duality theorem, it must be optimal.

(c) The dual problem is bounded if and only if the primal problem

$$\begin{aligned} \min \quad & [3,2,1,2,3]\,\mathbf{x} \\ \text{subject to} \quad & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\ & \mathbf{x} \geq 0 \end{aligned}$$

is feasible.

(d) Feasibility of the primal problem means that $\left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right]$ is a nonnegative linear combination of $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} 1 \\ 5 \end{smallmatrix}\right]$. (A picture reveals that) this is equivalent to $\left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right]$ being a nonnegative linear combination of $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ 5 \end{smallmatrix}\right]$, and that, in turn, is equivalent to $b_1 \leq b_2 \leq 5b_1$. $\square$

2. Consider a linear optimization problem in standard form

$$\begin{aligned} \min \quad & \mathbf{c} \cdot \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\,\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}\,. \end{aligned}$$

(a) Give a criterion that a basic solution $\mathbf{v}$ with corresponding basis matrix $\mathbf{B}$ is optimal.

(b) Now perturb the jth component of $\mathbf{b}$ by δ. Compute a condition that guarantees that the perturbed basic solution $\mathbf{v}$ is still optimal.

Solution.

(a) We need feasibility: $\widetilde{\mathbf{v}} = \mathbf{B}^{-1}\mathbf{b} \geq 0$, and optimality: the reduced cost vector $\mathbf{c} - \widetilde{\mathbf{c}}\,\mathbf{B}^{-1}\mathbf{A} \geq 0$.

(b) The reduced cost vector does not depend on $\mathbf{b}$ and so remains nonnegative. But we have to check feasibility, that is,

$$\mathbf{B}^{-1}\,(\mathbf{b} + \delta\,\mathbf{e}_j) \geq 0\,.$$

This can be restated, since $\mathbf{B}^{-1}(\mathbf{b} + \delta\,\mathbf{e}_j) = \widetilde{\mathbf{v}} + \delta\,\mathbf{B}^{-1}\mathbf{e}_j$, and $\mathbf{B}^{-1}\mathbf{e}_j$ is simply the jth column of $\mathbf{B}^{-1}$. Thus the perturbed $\mathbf{v}$ remains optimal if and only if $\widetilde{\mathbf{v}}$ plus the jth column of $\mathbf{B}^{-1}$ is nonnegative. $\square$

3. Consider the integer optimization problem

$$\begin{aligned} \min \quad & x_1 - 2\,x_2 \\ \text{subject to} \quad & \begin{bmatrix} -4 & 6 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 9 \\ 4 \end{bmatrix} \\ & \mathbf{x} \in \mathbb{Z}^4_{\geq 0}\,. \end{aligned}$$

(a) Compute an optimal solution and the optimal cost.

(b) Now suppose we need to minimize $x_1 - 2\,x_2$ subject to at least one (but not necessarily both) of the constraints

$$-4\,x_1 + 6\,x_2 \leq 9 \qquad \text{and} \qquad x_1 + x_2 \leq 4$$

to hold. Assuming that we know the absolute bounds

$$-4\,x_1 + 6\,x_2 \leq 100 \qquad \text{and} \qquad x_1 + x_2 \leq 100\,,$$

write a new integer optimization program that models this new problem. (You do not need to solve it.)

Solution.

(a) One can approach this graphically (interpreting x_3 and x_4 as slack variables), using the cutting-plane method, or the branch-and-bound algorithm. We show the latter.

The linear relaxation problem can be solved, e.g., using the tableau method; the optimal solution is $[\frac{3}{2}, \frac{5}{2}, 0, 0]$ with cost $-\frac{7}{2}$. We branch, say, with $x_2 \leq 2$ and $x_2 \geq 3$. The latter problem is infeasible; the former has optimal solution $[\frac{3}{4}, 2, 0, \frac{5}{4}]$ with cost $-\frac{13}{4}$. We branch this last problem with $x_1 \leq 0$ and $x_1 \geq 1$; the optimal solutions are $[0, \frac{3}{2}, 0, \frac{5}{2}]$ with cost -3, and $[1, 2, 1, 1]$ with cost -3, respectively. This means we can discard the branch $x_1 \leq 0$, and the optimal integer solution is $[1, 2, 1, 1]$ with cost -3.

(b) We introduce the binary variables y_1 and y_2, and modify the constraints to

$$-4\,x_1 + 6\,x_2 - 9 \leq (1 - y_1)(100 - 9) \qquad \text{and} \qquad x_1 + x_2 - 4 \leq (1 - y_2)(100 - 4)\,.$$

If, e.g., $y_1 = 0$ then this gives the (known) constraint $-4\,x_1 + 6\,x_2 \leq 100$, whereas $y_1 = 1$ gives $-4\,x_1 + 6\,x_2 \leq 9$. Thus we need to make sure that at least one of y_1 and y_2 is forced to be 1. This can be modeled as

$$\begin{aligned} \min \quad & x_1 - 2\,x_2 \\ \text{subject to} \quad & -4\,x_1 + 6\,x_2 + 91\,y_1 \leq 100 \\ & x_1 + x_2 + 96\,y_2 \leq 100 \\ & y_1 + y_2 \geq 1 \\ & 0 \leq y_1, y_2 \leq 1 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2, y_1, y_2 \in \mathbb{Z}\,. \end{aligned}$$

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