MATH 430 Midterm Exam 22 October 2018 9:00–9:50 a.m.

Name: _____

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. You may refer to theorems in the book and class notes.

1. Take a deep breath. You can do this!
   (a) Carefully define a *polyhedron* in $\mathbb{R}^n$.
   (b) Tell me (the inequality description of) your favorite polyhedron.
   (c) We studied three equivalent definitions: that of a *vertex*, an *extreme point*, and a *basic feasible solution*. Pick a vertex of your polyhedron and verify two of our three definitions for that vertex.

2. (a) Carefully define what we mean by a linear optimization problem *in standard form*.
   (b) Explain why every linear optimization problem in standard form has an optimal solution.
   (c) Convert the following problem into standard form:

$$\begin{aligned} \min \quad & 2x_1 + 3\,|x_2| \\ \text{subject to} \quad & |x_1 + 2| + x_2 \le 5\,. \end{aligned}$$

Solution.

(a) A linear optimization problem in standard form looks like

$$\begin{aligned} \min \quad & \mathbf{c} \cdot \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\,\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0}\,. \end{aligned}$$

(b) A polyhedron given by the constraints $\mathbf{A}\,\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$ contains no lines (because it is a subset of $\mathbb{R}^n_{\ge 0}$), and so it has at least one vertex. Since an optimization problem has an optimal solution at a vertex, and every polyhedron has only finitely many vertices, one of them must be optimal, or the optimal solution is $-\infty$.

(c) We first eliminate the use of absolute values:

$$\begin{aligned} \min \quad & 2x_1 + 3y \\ \text{subject to} \quad & x_1 + 2 + x_2 \le 5 \\ & -x_1 - 2 + x_2 \le 5 \\ & x_2 \le y \\ & -x_2 \le y\,. \end{aligned}$$

Next we adjust things so we only have nonnegative variables; note that $y \geq 0$ automatically:

$$\begin{aligned} \min \quad & 2(x_1^+ - x_1^-) + 3y \\ \text{subject to} \quad & x_1^+ - x_1^- + 2 + x_2^+ - x_2^- \leq 5 \\ & -x_1^+ + x_1^- - 2 + x_2^+ - x_2^- \leq 5 \\ & x_2^+ - x_2^- \leq y \\ & -x_2^+ + x_2^- \leq y \\ & x_1^+, x_1^-, x_2^+, x_2^-, y \geq 0\,. \end{aligned}$$

Finally, we introduce slack variables:

$$\begin{aligned} \min \quad & 2(x_1^+ - x_1^-) + 3y \\ \text{subject to} \quad & x_1^+ - x_1^- + 2 + x_2^+ - x_2^- + z_1 = 5 \\ & -x_1^+ + x_1^- - 2 + x_2^+ - x_2^- + z_2 = 5 \\ & x_2^+ - x_2^- - y + z_3 = 0 \\ & -x_2^+ + x_2^- - y + z_4 = 0 \\ & x_1^+, x_1^-, x_2^+, x_2^-, y, z_1, z_2, z_3, z_4 \geq 0\,. \end{aligned}$$

□

3. (a) Carefully define what it means for a basic feasible solution of a linear optimization problem to be *degenerate*.
   (b) Give an example of a linear optimization problem that has no degenerate basic feasible solutions.
   (c) Give an example of a linear optimization problem that has a degenerate basic feasible solution.

(One possible) Solution.

(a) For a linear optimization problem in $\mathbb{R}^n$, a basic feasible solution is degenerate if it has more than n active constraints.

(b) The linear optimization problem

$$\begin{aligned} \min \quad & x_1 \\ \text{subject to} \quad & x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

has the three vertices $[0,0]$, $[1,0]$, and $[0,1]$; none of them are degenerate.

(c) The linear optimization problem

$$\begin{aligned} \min \quad & x_1 \\ \text{subject to} \quad & x_1 + x_2 \leq 1 \\ & x_1 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

has the same three vertices $[0,0]$, $[1,0]$, and $[0,1]$; one of them $[1,0]$ is degenerate (with active constraints $x_1 + x_2 = 1$, $x_1 = 1$, and $x_2 = 0$). □

MATH 430 Take Home Midterm Exam 22–24 October 2018

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. You may refer to theorems in the text book or your class notes. You are welcome to use books and internet sources, but you are not allowed to discuss this exam with anyone (including your class mates).

1. Give an example of an unbounded polyhedron P.

   (a) Give an example of a cost vector $\mathbf{c}$ for which the linear optimization problem

   $$\begin{aligned} \min \quad & \mathbf{c} \cdot \mathbf{x} \\ \text{subject to} \quad & \mathbf{x} \in P \end{aligned}$$

   has optimal value $-\infty$.

   (b) Give an example of a cost vector $\mathbf{c}$ for which the linear optimization problem

   $$\begin{aligned} \min \quad & \mathbf{c} \cdot \mathbf{x} \\ \text{subject to} \quad & \mathbf{x} \in P \end{aligned}$$

   has a finite optimal solution.

   (One possible) Solution.

   (a) $P = [0, \infty)$

   (b) For $c = -1$ we obtain $cx = -x$ with an optimal value of $-\infty$ as $x \in P$.

   (c) For $c = 1$ we obtain $cx = x$ with an optimal value of 0 as $x \in P$. $\square$

2. Consider the linear optimization problem

   $$\begin{aligned} \min \quad & [3, 2, 1, 2, 3]\, \mathbf{x} \\ \text{subject to} \quad & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 7 \end{bmatrix} \\ & \mathbf{x} \geq 0\,. \end{aligned}$$

   (a) How many basic solutions are there?

   (b) How many basic feasible solutions are there?

   (c) Determine an optimal solution.

   Solution.

   (a) A basic solution necessarily has three variables equal to 0, and the basis matrix needs to be invertible. Since all possible basis matrices are invertible, there are $\binom{5}{2} = 10$ basic solutions.

(b) If the basic variables are x_a and x_b (for which we may assume $a < b$) then we are looking for the solution to

$$\begin{bmatrix} 1 & 1 \\ a & b \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix} \qquad \text{that is,} \qquad \begin{bmatrix} x_a \\ x_b \end{bmatrix} = \frac{1}{b-a} \begin{bmatrix} 2b-7 \\ 7-2a \end{bmatrix}$$

For these variables to be nonnegative, we need $b \geq \frac{7}{2}$ and $a \leq \frac{7}{2}$, so any of the combinations $a = 1, 2, 3$ and $b = 4, 5$ will do—there are 6 basic feasible solutions.

(c) The two optimal solutions are $[0, 0, 1, 1, 0]$ and $[0, 0, \frac{3}{2}, 0, \frac{1}{2}]$. □

3. Suppose we are given the linear optimization problem

$$\begin{aligned} \min \quad & \mathbf{c} \cdot \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\,\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}\,, \end{aligned}$$

and $\mathbf{v}$ is a feasible solution. Prove that $\mathbf{v}$ is optimal if and only if the linear optimization problem

$$(\star) \qquad \begin{aligned} \min \quad & \mathbf{c} \cdot \mathbf{d} \\ \text{subject to} \quad & \mathbf{A}\,\mathbf{d} = \mathbf{0} \\ & d_j \geq 0 \text{ for all } j \text{ such that } v_j = 0 \end{aligned}$$

has an optimal cost of 0.

Proof. By Homework 3.3, $\mathbf{d}$ is a feasible direction at $\mathbf{v}$ if and only if $\mathbf{A}\,\mathbf{d} = \mathbf{0}$ and $d_j \geq 0$ for all j such that $v_j = 0$. This means that the constraints $(\star)$ above describe precisely the set of feasible directions $\mathbf{d}$ at $\mathbf{v}$.

Thus $(\star)$ has an optimal cost of 0 if and only if $\mathbf{c} \cdot \mathbf{d} \geq 0$ for every feasible direction $\mathbf{d}$ at $\mathbf{v}$. By Homework 3.2, the latter condition holds if and only if $\mathbf{v}$ is optimal. □

4. Consider the linear optimization problem

$$\begin{aligned} \min \quad & -2x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 \leq 2 \\ & x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0\,. \end{aligned}$$

(a) Convert the problem into standard form and construct a basic feasible solution with $x_1 = x_2 = 0$.

(b) Run the simplex algorithm starting with this basic feasible solution.

(c) Draw a graphical representation of the problem in its original form and indicate the path taken by the simplex algorithm.

(One possible) Solution.

(a) By introducing two slack variables, the standard form becomes

$$\begin{aligned} \min \quad & [-2,-1,0,0]\,\mathbf{x} \\ \text{subject to} \quad & \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \\ & \mathbf{x} \geq 0\,. \end{aligned}$$

The basic feasible solution with $x_1 = x_2 = 0$ means $\mathbf{B}$ consists of the third and forth column—it is the identity matrix. Thus $\mathbf{B}^{-1}\left[\begin{smallmatrix}2\\6\end{smallmatrix}\right] = \left[\begin{smallmatrix}2\\6\end{smallmatrix}\right]$, and so the basic feasible solution is $[0,0,2,6]$.

(b) We compute in tableaux form; the pivot element is marked with $*$ in each step.

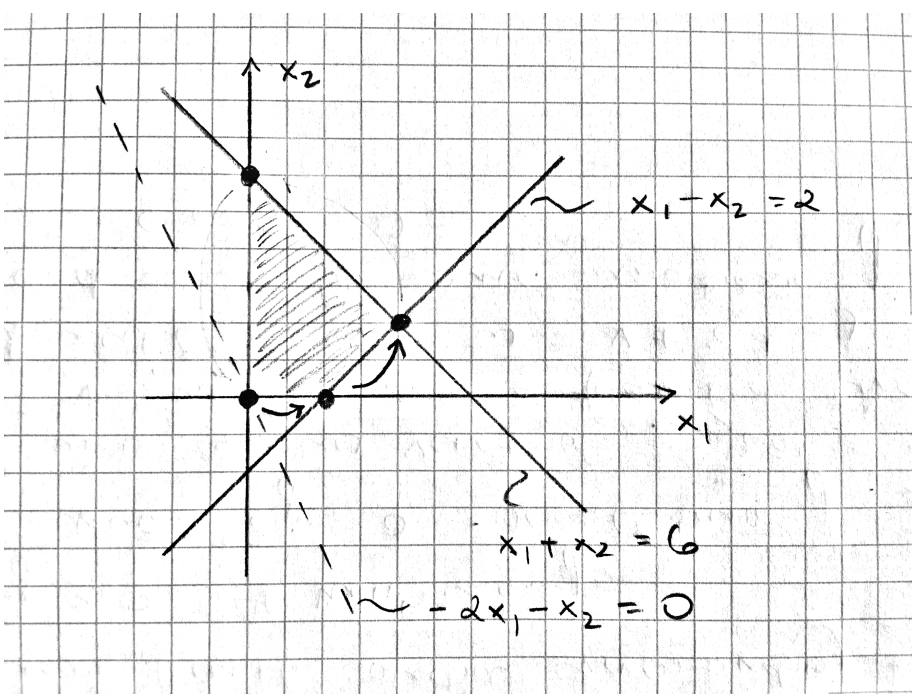
	0	-2	1	0	0
x_3	2	1^*	-1	1	0
x_4	6	1	1	0	0

	4	0	-1	2	0
x_1	2	1	-1^*	1	0
x_4	4	0	2	-1	1

	6	0	0	3/2	1/2
x_1	4	1	0	1/2	1/2
x_2	2	0	1	$-1/2$	1/2

Thus an optimal solution is $[4,2,0,0]$.

(c)



□