MATH 435/735 Midterm Exam 10/19/22 12:00–12:50 a.m.

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. You may refer to theorems in the book and class notes (without having to reference theorem numbers etc.).

1. (a) State one of the isomorphism theorems.
   (b) Suppose A and B are groups, and $C \trianglelefteq A$ and $D \trianglelefteq B$. Show that $C\times D \trianglelefteq A\times B$.
   (c) Prove that $(A \times B)/(C \times D) \simeq (A/C) \times (B/D)$.
   (*Hint:* there is a way to set up (c) so that (b) will follow as a side product.)

Proof.

(b) & (c) Let $\phi : A \times B \to (A/C) \times (B/D)$ be defined through $\phi(a, b) = (a\,C, b\,D)$. Then ϕ is a homomorphism and onto (both easy to check), and

$$\ker(\phi) \;=\; \{(a, b) \in A \times B : (a\,C, b\,D) = (C, D)\} \;=\; C \times D\,.$$

Thus $C\times D \trianglelefteq A\times B$, and by the First Isomorphism Theorem, $(A\times B)/(C\times D) \simeq (A/C) \times (B/D)$. ☐

2. Suppose the group G acts on a set A.
   (a) Given $a \in A$, define its orbit $\mathrm{orb}(a)$ and stabilizer G_a.
   (b) Prove that $|\mathrm{orb}(a)| = [G : G_a]$. (*Hint:* construct a bijection between $\mathrm{orb}(a)$ and the left cosets of G_a.)
   (c) Show that a group of order p^k, for some prime p, has a nontrivial center. (*Hint:* consider G acting on itself by conjugation and partitioning G into the orbits of this group action.)

Proof.

(b) Define $\phi : \mathrm{orb}(a) \to \{g\,G_a : g \in G\}$ by $\phi(g \cdot a) := g\,G_a$. This map is well defined and injective:

$$g\,G_a = h\,G_a \iff g^{-1}h \in G_a \iff \left(g^{-1}h\right) \cdot a = a$$
$$\iff g \cdot a = g\left(g^{-1}h\right) \cdot a = \left(gg^{-1}\right) h \cdot a = h \cdot a\,.$$

Since ϕ is clearly surjective, it is thus a bijection, and so there are $|\mathrm{orb}(a)|$ left cosets of G_a, whose number is also $[G : G_a]$.

(c) The stabilizer of $a \in G$ under conjugation is the centralizer $C_G(a)$, and $\mathrm{orb}(a) = \{a\}$ if and only if $a \in Z(G)$. In particular, if $a \notin Z(G)$ then $|\mathrm{orb}(a)| = [G : C_G(a)]$ is divisible by p. We know that we can partition G into its orbits, from which we can write $|G|$ as the number of single-element orbits (which is $|Z(G)|$) plus some number divisible by p. Because $|G|$ is also divisible by p, so is $|Z(G)|$. ☐

3. (a) State one of the three parts of Sylow's Theorem.
   (b) Find all Sylow 2-subgroups of A_4 and determine their isomorphism type.
   (c) Find all Sylow 3-subgroups of A_4 and determine their isomorphism type.

   Proof.

   (b) There is only one subgroup of A_4 of order 4, namely $\langle (12)(34), (13)(24) \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
   (c) There are four subgroups of A_4 of order 3, namely those generated by a 3-cycle; all are isomorphic to $\mathbb{Z}_3$. □

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Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. You may refer to theorems in the text book. You are welcome to use books and internet sources, but you are not allowed to discuss this exam with anyone (including your class mates).

If you're enrolled in MATH 435, submit only 3 of the following 4 problems.

1. Let $G \subset \mathbb{R}^{2\times 2}$ consists of all (2×2)-matrices that have exactly one nonzero entry in each row and column, and this entry is ± 1. Show that G is a group under matrix multiplication, and that $G \cong D_8$.

   (Bonus question) Realize G as a semidirect product isomorphic to $\mathbb{Z}_2^2 \rtimes S_2$ and conclude that D_8 is isomorphic to both

$$\mathbb{Z}_4 \rtimes_\phi \mathbb{Z}_2 \qquad \text{and} \qquad (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_\psi \mathbb{Z}_2$$

   for some homomorphisms ϕ and ψ (even though $\mathbb{Z}_4 \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$).

   Proof. There are 8 elements in G, and by listing them we can see that $G = \langle r, s \rangle$ where

$$r := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad s := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   Since r has order 4, s has order 2, and $rsrs$ is the identity matrix, we know that $G \cong D_8$. □

   Bonus question. We consider $\{\pm 1\}$ as a group under multiplication, isomorphic to $\mathbb{Z}_2$. Similar to the octahedral group discussed in class, let S_2 act on $\mathbb{Z}_2^2$ by permuting the coordinates of $\binom{x}{y} \in \mathbb{Z}_2^2$, giving rise to a homomorphism $\psi : S_2 \to \mathbb{Z}_2^2$. We easily check that the elements

$$\tilde{r} := \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, (12) \right) \qquad \text{and} \qquad \tilde{s} := \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{id} \right)$$

in $\mathbb{Z}_2^2 \rtimes_\psi S_2$ have order 4 and 2, respectively, and that $\tilde{r}\tilde{s}\tilde{r}\tilde{s} = \big(\binom{1}{1}, \mathrm{id}\big)$. Thus $G \cong \mathbb{Z}_2^2 \rtimes_\psi S_2$ via an isomorphism defined on the respective generators by $r \mapsto \tilde{r}$ and $s \mapsto \tilde{s}$. Since $S_2 \cong \mathbb{Z}_2$, we have proved that

$$D_8 \;\cong\; (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_\psi \mathbb{Z}_2 \;\cong\; \mathbb{Z}_4 \rtimes_\phi \mathbb{Z}_2$$

where ϕ was given in class (here $\mathbb{Z}_2$ acts on $\mathbb{Z}_4$ by inversion). □

2. Show that $\mathrm{GL}_2(\mathbb{R})$, the group of invertible (2×2) real matrices, has a normal subgroup H such that the quotient group $\mathrm{GL}_2(\mathbb{R})/H$ is isomorphic to $\mathbb{R}^*$, the multiplicative group of nonzero real numbers.

   Proof. Let $\phi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^*$ be given by

   $$\phi \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;:=\; ad - bc\,,$$

   i.e., $\phi(A) = \det(A)$. This map is well defined because matrices in $\mathrm{GL}_2(\mathbb{R})$ have nonzero determinant, and ϕ is a homomorphism because the determinant is multiplicative. Furthermore, ϕ is surjective: for any $r \in \mathbb{R}_{>0}$,

   $$\phi \begin{bmatrix} \sqrt{r} & 0 \\ 0 & \pm\sqrt{r} \end{bmatrix} \;=\; \pm r\,.$$

   By the First Isomorphism Theorem, $\mathrm{GL}_2(\mathbb{R})/\ker(\phi) \cong \mathbb{R}^*$ and $\ker(\phi) \trianglelefteq \mathrm{GL}_2(\mathbb{R})$. □

3. Prove that any group of order $20677 = 23 \cdot 29 \cdot 31$ is cyclic.

   Proof. Suppose $|G| = 20677 = 23 \cdot 29 \cdot 31$. Sylow's theorem say that $n_{23} | (29 \cdot 31)$ (which gives the possibilities $n_{23} = 1$, 29, 31, or 899) and $n_{23} \equiv 1 \bmod 23$, which implies that there is a unique Sylow 23-subgroup $H \leq G$. Similar arguments imply that there is a unique Sylow 29-subgroup $J \leq G$ and a unique Sylow 31-subgroup $K \leq G$.

   The subgroup H is normal: for any $g \in G$, gHg^{-1} is also of order 23, and so because H is the unique subgroup of G or this order, $gHg^{-1} = H$. The same argument shows that J and K are normal.

   Thus HJ is a subgroup of G, and since $H \cap J$ is a subgroup of both $H \cong \mathbb{Z}_{23}$ and $J \cong \mathbb{Z}_{29}$, we have $H \cap J = \{e\}$, and so $HJ \cong H \times J \cong Z_{667}$. Repeating the argument for HJ and K gives

   $$G \;=\; HJK \;\cong\; H \times J \times K \;\cong\; Z_{20677}\,.$$ □

4. Suppose that G is a finite Abelian group and G has no element of order 2. Show that the mapping $g \mapsto g^2$ is an automorphism of G. Show, by example, that if G is infinite the mapping need not be an automorphism.

   Proof. Let $\phi : G \to G$ with $\phi(g) = g^2$. Because G is Abelian, $\phi(gh) = (gh)^2 = g^2h^2$, and thus ϕ is a homomorphism. The kernel of ϕ is trivial (because an element in the kernel would have order 2), and so ϕ is injective. Because G is finite, this implies that ϕ is also surjective.

   For the additive group $G = \mathbb{Z}$ (which has no elements of order 2), the mapping is $x \mapsto 2x$, which is not a surjection. □