MATH 435/735 Module Homework I

(1) Suppose R is a ring. Prove that is M and N are free R-modules, then $M \oplus N$ is also a free R-module.

(2) Suppose R is a ring and M is an R-module. Show that a set that contains a torsion element of M cannot be a basis of M.

(3) Let R be a commutative ring, viewed as an R-module, and let I be an ideal of R. Show that any two elements in I are linearly dependent.

(4) (Grad students) Let R be a commutative ring. Prove that $R^m \cong R^n$ if and only if $m = n$. (*Hint:* try the same trick as in Monday's lecture—start with a maximal ideal of R to reduce the problem to that of vector spaces.)

(5) Consider the $\mathbb{Z}$-module $M := \mathbb{Z} \times \mathbb{Z} \times \cdots$ (which you may think of as consisting of all integer sequences). Let R be the set of all homomorphisms $M \to M$. This set R becomes a (non-commutative) ring, as usual, by defining for $f, g \in R$

$$(f+g)(x) := f(x) + g(x) \qquad \text{and} \qquad (fg)(x) := f(g(x)) .$$

Now consider R as an R-module.

(a) Define the functions $f_1, f_2, g_1, g_2 \in R$ by

$$\begin{aligned} f_1(x_1, x_2, \dots) &:= (x_1, x_3, x_5, \dots) \\ f_2(x_1, x_2, \dots) &:= (x_2, x_4, x_6, \dots) \\ g_1(x_1, x_2, \dots) &:= (x_1, 0, x_2, 0, x_3, \dots) \\ g_2(x_1, x_2, \dots) &:= (0, x_1, 0, x_2, 0, x_3, \dots) . \end{aligned}$$

Show that $f_1 g_1 = f_2 g_2 = 1$, $f_1 g_2 = f_2 g_1 = 0$, and $g_1 f_1 + g_2 f_2 = 1$.

(b) Show that $\{f_1, f_2\}$ is a basis of R.

(c) Show that $R \cong R^2$.

(This gives an example that free modules over non-commutative rings need not have a well-defined notion of rank/dimension.)