MATH 725 Final Exam (& Solutions) 12/15/21

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. You may refer to theorems in the text book.

You are welcome to use books and internet sources, but you are not allowed to discuss this exam with anyone (this includes live discussions, calls, chats, etc.). I reserve the right for an follow-up oral exam if I suspect that you did not follow these rules.

The take-home exam is due on at 12:00 p.m. on 17 December 2021 (via email), and your submission should be a pdf file (typed or carefully scanned).

(1) Consider the vector space $\mathscr{P}_4(\mathbf{C})$ of polynomials of degree ≤ 4 and the linear function $D : \mathscr{P}_4(\mathbf{C}) \to \mathscr{P}_4(\mathbf{C})$ given by $D(p(x)) := p'(x)$.

(a) Determine all generalized eigenspaces and the Jordan Normal Form of D.

(b) Compute the minimal and characteristic polynomial of D.

Proof. (a) Since $D^5 = 0$, we have $\operatorname{null}(D^5) = \mathscr{P}_4(\mathbf{C})$, and so 0 is the only eigenvalue (with generalized eigenspace $\mathscr{P}_4(\mathbf{C})$). Furthermore, the only eigenvectors are the constant polynomials, and so the Jordan Normal Form of D is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(b) Since 0 is the only eigenvalue of D, its characteristic polynomial is x^5. The minimal polynomial divides the characteristic polynomial, and since $p(x) = x^4$ does not vanish after differentiating four times, the minimal polynomial is also x^5. □

(2) Let V be a complex vector space.

(a) Determine all linear functions $g : V \to V$ for which each $v \in V \setminus \{0\}$ is an eigenvector. (*Hint:* consider distinct eigenvalues of g.)

(b) Give an example of a complex vector space and a nonzero linear function $f : V \to V$ for which each $\lambda \in \mathbf{C}$ is an eigenvalue.

Proof. (a) We claim that g has only one eigenvalue λ, and thus $g(v) = \lambda\, v$. To prove the claim, suppose λ and μ are distinct eigenvalues, with eigenvectors v and w, respectively. Then, as we proved in class, v and w are linearly independent. However, $v + w$ is, by assumption, also an eigenvector, with some eigenvalue κ, and so

$$0 \;=\; f(v) + f(w) - f(v+w) \;=\; \lambda\, v + \mu\, w - \kappa(v+w) \;=\; (\lambda - \kappa)v + (\mu - \kappa)w\,.$$

Thus $\lambda = \kappa = \mu$, a contradiction.

(b) Let V be the vector space of all complex sequences and

$$f(x_1, x_2, x_3, \ldots) := (x_2, x_3, x_4, \ldots).$$

Then for any $\lambda \in \mathbf{C}$,

$$f\left(1, \lambda, \lambda^2, \ldots\right) = \left(\lambda, \lambda^2, \lambda^3, \ldots\right) = \lambda\left(1, \lambda, \lambda^2, \ldots\right)$$

and so (since $\left(1, \lambda, \lambda^2, \ldots\right) \neq 0$, even when $\lambda = 0$) λ is an eigenvalue. $\square$

(3) Suppose $A = (a_{jk}) \in \mathbf{C}^{n \times n}$ and let

$$||A|| := \sqrt{\sum_{j,k=1}^{n} \left|a_{jk}\right|^2}.$$

Prove that

$$||A||^2 = s_1^2 + s_2^2 + \cdots + s_n^2,$$

where $s_1, s_2, \ldots, s_n$ are the singular values of A. (*Hint:* start by showing that $||A||^2$ equals the trace of A^*A.)

Remark: this matrix norm is called the *Frobenius norm*.

Proof. We start with proving the hint:

$$\operatorname{tr}(A^*A) = \sum_{k=1}^{n} (A^*A)_{kk} = \sum_{k=1}^{n} \sum_{j=1}^{n} \overline{a_{jk}}\, a_{jk} = ||A||^2.$$

Singular Value Decomposition says that there exist orthonormal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ such that

$$A v = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n.$$

We can express this in matrix form as follows. Let E be the matrix formed by $e_1, \ldots, e_n$ as column vectors, let F be the matrix formed by $f_1, \ldots, f_n$ as column vectors, and let S be a diagonal matrix with $s_1, \ldots, s_n$ on the diagonal. Note that $E^{-1} = E^*$ and $F^{-1} = F^*$ because each basis is orthonormal. Now $A\, e_k = s_k\, f_k$ gives

$$AE = FS,$$

i.e., $A = FSE^*$. Thus (using the hint)

$$||A||^2 = \operatorname{tr}(A^*A) = \operatorname{tr}(ES^*F^*FSE^*) = \operatorname{tr}\left(S^2\right) = s_1^2 + s_2^2 + \cdots + s_n^2.$$

Here we have used that $\operatorname{tr}(MN) = \operatorname{tr}(NM)$ and that the singular values are real numbers. $\square$

(4) (a) Suppose $x_1, x_2, \ldots, x_n \in \mathbf{C}$, and let $A : \mathbf{C}^n \to \mathbf{C}^n$ be given in matrix form (with respect to the standard basis of $\mathbf{C}^n$)

$$A := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}.$$

Viewing $x_1, x_2, \dots, x_n$ as variables, prove that $\det(A)$ is a polynomial in $x_1, x_2, \dots, x_n$ of (total) degree at most $\frac{n(n-1)}{2}$.

(b) Show that $\det(A) = 0$ if $x_j = x_k$ for some $j \neq k$, and conclude that $x_k - x_j$ divides $\det(A)$.

(c) Prove that

$$\det(A) = \prod_{1 \leq j < k \leq n} \left(x_k - x_j\right).$$

(*Hint:* use (a) and (b) to show that $\det(A) = c \prod_{1 \leq j < k \leq n} \left(x_k - x_j\right)$ for some constant c, and then compute the coefficient of $x_1^0 x_2^1 \cdots x_n^{n-1}$ on both sides.)

Proof. (a) The determinant formula for a matrix we proved in class gives

$$\det(A) = \sum_{\pi \in S_n} \operatorname{sign}(\pi) \prod_{j=1}^{n} x_{\pi(j)}^{j-1}$$

which is a polynomial of degree at most $\sum_{j=1}^{n} (j-1) = \frac{n(n-1)}{2}$.

(b) If $x_j = x_k$ for some $j \neq k$ then two rows of A are equal, in which case we know that $\det(A) = 0$. Viewing $\det(A)$ as a polynomial in x_k, this means that x_j is a root, and so $x_k - x_j$ divides $\det(A)$.

(c) From part (b) we know that $\prod_{1 \leq j < k \leq n} \left(x_k - x_j\right)$, which is a polynomial of degree $\frac{n(n-1)}{2}$, divides $\det(A)$. Part (a) then implies that the degree of $\det(A)$ must equal $\frac{n(n-1)}{2}$, and so

$$\det(A) = c \prod_{1 \leq j < k \leq n} \left(x_k - x_j\right)$$

for some constant c. The coefficient of $x_1^0 x_2^1 \cdots x_n^{n-1}$ in $\det(A)$ is $\operatorname{sign}(\mathrm{I}) = 1$, as is the coefficient of $x_1^0 x_2^1 \cdots x_n^{n-1}$ in $\prod_{1 \leq j < k \leq n} \left(x_k - x_j\right)$, and so $c = 1$. $\square$

Remark: We have just computed the famous *Vandermonte determinant*.