MATH 725 Midterm Exam

Part I (in-class exam, 20 October 2021, 9:30–10:45 a.m.)

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. As usual, **F** *stands for either* **R** *or* **C***.*

(1) Suppose V and W are vector spaces.
  (a) Define what it means for a set $S \subseteq V$ to be a *basis* of V.
  (b) Define the *dimension* of V.
  (c) Recall the definition of the binomial coefficient
$$\binom{x}{n} := \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}$$
  where $n \geq 0$ is an integer and we may view x as a variable. (We set $\binom{x}{0} = 1$.) Show that $\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{n}$ form a basis of $\mathscr{P}_n(\mathbf{F})$, the set of all polynomials of degree $\leq n$ with coefficients in $\mathbf{F}$.

Solution for (c). Suppose
$$\lambda_0 + \lambda_1 \binom{x}{1} + \cdots + \lambda_n \binom{x}{n} \; = \; 0 .$$
Plugging in $x = 0$ yields $\lambda_0 = 0$. But then plugging in $x = 1$ yields $\lambda_1 = 0$. We continue this process, proving that each $\lambda_j = 0$. □

(2) Suppose V and W are vector spaces.
  (a) Define what it means for a map $f : V \to W$ to be *linear*.
  (b) Define the *null space* and the *range* of f.
  (c) Give an example of a linear map that has a two-dimensional null space and a three-dimensional range.

Solution for (c). Let $f \in L\left(\mathbf{R}^5\right)$ given by the matrix
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$
□

(3) Suppose V is a vector space and $f \in L(V)$.
  (a) Define the notion of *eigenvalue* and *eigenvector* of f.
  (b) Give an example of a linear map on a real vector space that has no eigenvalues.
  (c) State a condition on V or f that guarantees that f has an eigenvalue.

Solution. (b) Let $f \in L\left(\mathbf{R}^2\right)$ given by the matrix
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$
(c) V is a complex vector space. (Alternative: V has a basis with respect to which f is upper triangular.) □

(4) Suppose V is a vector space over $\mathbf{F}$.
  (a) Define the notion of an *inner product* for V.

(b) Now consider $V = \mathbf{C}[x]$ with $\langle f,g\rangle := \int_{-1}^{1} f(x)\overline{g(x)}\,dx$. Show that this defines an inner-product space.
(c) Compute the norm of x^n, where n is a nonnegative integer.

Solution. (b) First, $\langle f,f\rangle = \int_{-1}^{1} f(x)\overline{f(x)}\,dx = \int_{-1}^{1} |f(x)|^2\,dx$, and so this real integral over a nonnegative function is ≥ 0 and equals 0 if and only if the integrand is the zero function (which is equivalent to f being the zero function).
Second, $\langle af_1+f_2,g\rangle = \int_{-1}^{1} (a f_1(x)+f_2(x))\,\overline{g(x)}\,dx = a\int_{-1}^{1} f_1\overline{g(x)}\,dx + \int_{-1}^{1} f_2\overline{g(x)}\,dx$.
Third, $\langle f,g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}\,dx = \int_{-1}^{1} \overline{\overline{f(x)}g(x)}\,dx = \overline{\int_{-1}^{1} \overline{f(x)}g(x)\,dx} = \overline{\langle g,f\rangle}$.
(c)

$$||x^n|| = \sqrt{\langle x^n,x^n\rangle} = \sqrt{\int_{-1}^{1} |x^n|^2\,dx} = \sqrt{\int_{-1}^{1} |x|^{2n}dx} = \sqrt{2\int_{0}^{1} x^{2n}dx} = \sqrt{\frac{2}{2n+1}}\,. \qquad \square$$

Part II (take-home exam)

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. You may refer to theorems in the text book. As usual, **F** *stands for either* **R** *or* **C***.*

You are welcome to use books and internet sources, but you are not allowed to discuss this exam with anyone (this includes live discussions, calls, chats, etc.). I reserve the right for an oral follow-up exam if I suspect that you did not follow these rules.

The take-home exam is due on at 9:30 a.m. on 10 October 2021 (via email), and your submission should be a pdf file (typed or carefully scanned).

(1) Let M be the vector space of all real $n\times n$ matrices, for some fixed $n\in\mathbf{Z}_{>0}$. For $A = \left(a_{jk}\right)\in M$, define the *trace* of A as

$$\mathrm{tr}(A) := \sum_{j=1}^{n} a_{jj}\,.$$

(a) Show that $U := \{A\in M : \mathrm{tr}(A) = 0\}$ is a subspace of M.
(b) Compute the dimension of U.

Proof. (a) The zero matrix has trace zero, and for $A = \left(a_{jk}\right)$, $B = \left(b_{jk}\right)$, and $r\in\mathbf{R}$, then

$$\mathrm{tr}(rA+B) = \sum_{j=1}^{n}\left(r\,a_{jj}+b_{jj}\right) = r\sum_{j=1}^{n} a_{jj} + \sum_{j=1}^{n} b_{jj} = r\,\mathrm{tr}(A)+\mathrm{tr}(B)\,, \qquad (\star)$$

and so if $\mathrm{tr}(A) = \mathrm{tr}(B) = 0$, we have $\mathrm{tr}(rA+B) = 0$.
(b) Considering the trace as a map $\mathrm{tr}: M\to\mathbf{R}$, we showed in $(\star)$ that tr is linear. But U is, by definition, the null space of tr. Since tr is surjective (we can reach any $r\in\mathbf{R}$ by considering a matrix with $a_{11} = r$ and all other entries 0), we have

$$n^2 = \dim M = \dim\mathrm{null}(\mathrm{tr}) + \mathrm{range}(\mathrm{tr}) = \dim U + 1\,,$$

and thus $\dim U = n^2-1$. $\square$

(2) As usual, let $\mathbf{R}[x]$ be the vector space of all polynomials with coefficients in $\mathbf{R}$.[1]

[1] In this exercise, you may freely cite theorems from Calculus.

(a) Show that $\frac{d}{dx}$ is a linear map $\mathbf{R}[x] \to \mathbf{R}[x]$. Is the map injective or surjective or both?

(b) Fix $a \in \mathbf{R}$ and let $I_a : \mathbf{R}[x] \to \mathbf{R}[x]$ be defined by $I_a(f) := \int_a^x f(t)\, dt$. Show that I_a is linear. Is I_a injective or surjective or both?

(c) Is it possible to choose a value of a so that I_a is the inverse of $\frac{d}{dx}$? Explain.

Proof. (a) Given $p(x), q(x) \in \mathbf{R}[x]$ and $\lambda \in \mathbf{R}$,

$$\tfrac{d}{dx} (\lambda\, p(x) + q(x)) = \lambda\, p'(x) + q'(x)$$

by the rules of calculus, so $\frac{d}{dx}$ is linear. This map is surjective because by the Fundamental Theorem of Calculus

$$\frac{d}{dx} \int_0^x p(t)\, dt = p(x)\,,$$

i.e., the polynomial $\int_0^x p(t)\, dt$ is a pre-image of $p(x)$. Differentiation is not injective because $\frac{d}{dx}(x+1) = \frac{d}{dx}(x+2) = 1$.

(b) Given $p(x), q(x) \in \mathbf{R}[x]$ and $\lambda \in \mathbf{R}$,

$$I_a(\lambda\, p(x) + q(x)) = \int_a^x \lambda\, p(t) + q(t)\, dt = \lambda \int_a^x p(t)\, dt + \int_a^x q(t)\, dt$$

by the rules of calculus, so I_a is linear. This map is injective because if

$$\int_a^x p(t)\, dt = \int_a^x q(t)\, dt$$

then we can differentiate both sides to conclude $p(x) = q(x)$. The map I_a is not surjective, because $I_a(p(x))$ is a polynomial of degree ≥ 1 (unless it is zero).

(c) The Fundamental Theorem of Calculus says that I_a is a right inverse of $\frac{d}{dx}$. It cannot be a two-sided inverse because then it would be surjective. □

(3) Let V be vector space over $\mathbf{F}$, and let $f \in L(V)$. Suppose there exists $k \in \mathbf{Z}_{>0}$ such that $f^k = 0$.[2] Prove that 0 is the only eigenvalue of f.

Proof. Suppose $f^k = 0$ and k is the smallest positive integer with this property.[3] If $k = 1$, f is the zero map, which certainly has 0 as an eigenvalue. If $k > 1$, then there exists $\mathbf{v} \in V$ such that $f^{k-1}(\mathbf{v}) \neq \mathbf{0}$, and so (because $f\left(f^{k-1}(\mathbf{v})\right) = \mathbf{0} = 0\,\mathbf{v}$) 0 is an eigenvalue of f.

Now suppose λ is another eigenvalue with eigenvector $\mathbf{v} \neq \mathbf{0}$. Then $\mathbf{0} = f^k(\mathbf{v}) = \lambda^k\, \mathbf{v}$ and so $\lambda = 0$.[4] □

(4) Consider the vector space $\mathscr{P}_2(\mathbf{C})$ of polynomials of degree ≤ 2, equipped with the inner product $\langle f, g\rangle := \int_{-1}^1 f(x)\overline{g(x)}\, dx$. Compute the orthogonal complement of $\mathscr{P}_1(\mathbf{C})$.

Proof. We already computed $||x^n|| = \sqrt{\frac{2}{2n+1}}$. We now apply "Gram–Schmidt" to the basis $\left(1, x, x^2\right)$ of $\mathscr{P}_2(\mathbf{C})$. Since $||1|| = \sqrt{2}$, the first basis vector (polynomial) is $\mathbf{e}_1 = \frac{1}{\sqrt{2}}$. To compute $\mathbf{e}_2$, we calculate

$$x - \langle x, \mathbf{e}_1\rangle\, \mathbf{e}_1 = x - \frac{1}{2} \int_{-1}^1 x\, dx = x$$

[2] Here 0 is the linear operator that returns $\mathbf{0}$ for every input vector.

[3] A linear operator $f \in L(V)$ with this property is *nilpotent*.

[4] Here it is important that we are working over $\mathbf{R}$ or $\mathbf{C}$, since otherwise we cannot conclude from $\lambda^k = 0$ that $\lambda = 0$.

and so $\mathbf{e}_2 = \frac{x}{||x||} = \sqrt{\frac{3}{2}}\,x$. To compute $\mathbf{e}_3$,

$$x^2 - \left\langle x^2, \mathbf{e}_1 \right\rangle \mathbf{e}_1 - \left\langle x^2, \mathbf{e}_2 \right\rangle \mathbf{e}_2 = x^2 - \frac{1}{2}\int_{-1}^{1} x^2\,dx - \frac{3}{2}x\int_{-1}^{1} x^3\,dx = x^2 - \frac{1}{3}$$

and so $\mathbf{e}_3 = \frac{x^2-\frac{1}{3}}{\left|\left|x^2-\frac{1}{3}\right|\right|} = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$.[5] Since $\mathbf{e}_1$ and $\mathbf{e}_2$ span $\mathscr{P}_1(\mathbf{C})$,

$$\mathscr{P}_1(\mathbf{C})^{\perp} \;=\; \left\{c\left(x^2 - \tfrac{1}{3}\right) : c \in \mathbf{C}\right\}. \qquad \square$$

[5] We have just computed the first three *Legendre polynomials*.