MATH 850 Combinatorial Commutative Algebra Homework

(1) Let Δ be a simplicial complex with corresponding Stanley–Reisner ideal I_Δ, and let

$$m^\tau \;:=\; \langle x_j : j \in \tau \rangle \,,$$

the monomial (prime) ideal corresponding to $\tau \subseteq [n]$. Show that

$$I_\Delta \;=\; \bigcap_{\sigma \in \Delta} m^{[n] \setminus \sigma}.$$

(2) Let $R := \mathbb{F}[x_1, x_2, x_3, x_4]$ and $I := \langle x_1, x_2, x_3, x_4 \rangle$. Compute a finite free resolution for the R-module R/I.

(3) Let Δ be the boundary of a pentagon. Compute I_Δ and one of its finite free resolutions.

(4) Let Δ be the boundary of an octahedron. Compute I_Δ and one of its finite free resolutions.

(5) Let $I := \langle x_1x_3, x_1x_4, x_2x_4 \rangle \subset \mathbb{F}[x_1, x_2, x_3, x_4]$, and let I_d denote the $\mathbb{F}$-vector space of homogeneous polynomials in I of degree d. Compute the *Hilbert function* $h_I(n) := \dim_{\mathbb{F}}(I_n)$ and the *Hilbert series*

$$H_I(x) \;:=\; \sum_{n \geq 0} h_I(n)\, x^n.$$

(6) Compute the Hilbert series of $\mathbb{F}[x_1, x_2, \ldots, x_5]/I_\Delta$ for Δ be the boundary of a pentagon, and verify that it yields the correct face numbers.

(7) Compute the Hilbert series of $\mathbb{F}[x_1, x_2, \ldots, x_6]/I_\Delta$ for Δ be the boundary of an octahedron, and verify that it yields the correct face numbers.

(8) Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$. Prove that the following are equivalent:

(a) There exists a polynomial $p(x)$ of degree d such that $f(n) = p(n)$ for sufficiently large integers n.

(b) There exists a polynomial $g(x)$ such that

$$\sum_{n \geq 0} f(n)\, x^n \;=\; \frac{g(x)}{(1-x)^{d+1}} \,.$$

(9) Let Δ_1 and Δ_2 be simplicial complexes on the disjoint sets E_1 and E_2. The *join* $\Delta_1 \star \Delta_2$ is the simplicial complex on $E_1 \cup E_2$ whose faces are the sets $\sigma_1 \cup \sigma_2$ for $\sigma_1 \in \Delta_1$ and $\sigma_2 \in \Delta_2$. Compute the h-vector of $\Delta_1 \star \Delta_2$ in terms of the h-vectors for Δ_1 and Δ_2.

(10) Find the open software system `Macaulay2` and figure out how to compute free resolutions with it. Go play.