MATH 850 Gröbner Bases Homework

(1) Divide x^2y+1 and x^3+y^2 into $1+x^5+x+y+x^3y+x^4y+y^2+2x^2y^2+xy^3$ twice, giving preference for one or the other polynomial. Use lex order with $x \succ y$.

(2) Let R be a ring, $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in R$, and let F be a field.

  (a) Show that $a_1a_2\cdots a_n - b_1b_2\cdots b_n \in \langle a_1-b_1, a_2-b_2, \ldots, a_n-b_n\rangle$.

  (b) Prove that, if $f(\mathbf{x}), f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}) \in F[\mathbf{x}]$ and $f = p(f_1, f_2, \ldots, f_m)$ for some $p(\mathbf{y}) \in F[\mathbf{y}]$, then

$$f(\mathbf{x}) - p(\mathbf{y}) \;\in\; \langle y_1 - f_1, y_2 - f_2, \ldots, y_m - f_m\rangle$$

  in $F[\mathbf{x}, \mathbf{y}]$.

(3) Let $G = \left(x^2+y,\, x^2y+1\right) \subseteq \mathbb{R}[x,y]$. Show that G is not a Gröbner basis with respect to any term order on $\mathbb{Z}^2_{\geq 0}$.

(4) Let $I = \left\langle x^2+y^2+z^2-4,\, x^2+2y^2-5,\, xz-1\right\rangle \subseteq \mathbb{R}[x,y,z]$. Compute a Gröbner basis for I with respect to lex orders with $x \succ y \succ z$ and $y \succ z \succ x$, and graded lex order.

(5) Compute a polynomial $p(y_1, y_2)$ such that $x_1^4 + x_2^4 = p(x_1+x_2,\, x_1x_2)$.

(6) Let $I = \langle 5x+y+z-17,\, x+y-z-1,\, x+y+z-9\rangle \subseteq \mathbb{R}[x,y,z]$. Compute a Gröbner basis for I with respect to the usual lex order. Explain the relation to Gauß elimination when solving the following system:

$$\begin{aligned} 5x+y+z &= 17 \\ x+y-z &= 1 \\ x+y+z &= 9\,. \end{aligned}$$

(7) Blow the dust off your old calculus book and convince yourself that, to find the minimum and maximum values of

$$x^3 + 2xyz - z^2$$

for (x,y,z) on the sphere given by $x^2+y^2+z^2=1$ can be solved using Lagrange multipliers via the system

$$\begin{aligned} 3x^2 + 2yz - 2x\lambda &= 0 \\ 2xz - 2y\lambda &= 0 \\ 2xy - 2z - 2z\lambda &= 0 \\ x^2+y^2+z^2-1 &= 0\,. \end{aligned}$$

Solve this via Gröbner bases; we suggest to use lex order with $\lambda \succ x \succ y \succ z$.