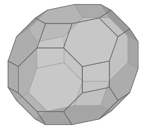
Polytopes & Varieties
Spring 2022



Homework I
due 4 February

(1) For the time being[1], let's define the *vertices* of a V-polytope as follows: Given $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbf{R}^d$, let $P := \operatorname{conv}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Then $\mathbf{v}_j$ is *not* a vertex of $P$ if $P = \operatorname{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \mathbf{v}_n\}$; all other $\mathbf{v}_j$'s are vertices of $P$.
  (a) Let $f : \mathbf{R}^d \to \mathbf{R}^k$ be an affine map. Show that $f(P)$ is convex.
  (b) What are the vertices of $f(P)$? (*Caveat:* This question has, in some cases, only a partial answer.)

(2) Let $P$ and $Q$ be V-polytopes in $\mathbf{R}^1$, with vertices $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{q}_1, \mathbf{q}_2$, respectively.
  (a) Define an affine map such that $f(P) = Q$.
  (b) Now assume $P$ and $Q$ are still 1-dimensional but live in $\mathbf{R}^d$ and $\mathbf{R}^k$, respectively. Define an affine map such that $f(P) = Q$.

(3) (**Collaborative**) Do one of the following problems:
  (a) Define a `SageMath` function that implements (2b), that is, takes as input two points in $\mathbf{R}^d$ (the vertices of $P$) and two points in $\mathbf{R}^k$ (the vertices of $Q$) and outputs the matrix and translation vector defining the affine map $f$ that takes $P$ to $Q$.
  (b) Find a nontrivial[2] polytope on `www.polytopia.eu` and realize it.[3]

(4) Consider the following optimization problem:

$$\begin{aligned} \min \quad & -x_1 - x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \le 3 \\ & 2x_1 + x_2 \le 3 \\ & x_1, x_2 \ge 0\,. \end{aligned}$$

  (a) Compute the V-description of the feasible region.
  (b) Solve the optimization problem.

[1] We'll have a different (more robust) definition of *vertex* soon.

[2] Let's say *nontrivial* means at least one vertex is on 4 edges and at least one face is not a triangle.

[3] This means finding vertex coordinates so that your polytope has the same combinatorics, i.e., isomorphic vertex-edge adjacencies. (Trust your intuition of what *same combinatorics* means, and trust that this intuition will be challenged in dimensions $> 3$.)