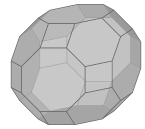
MATH 883: POLYTOPES & VARIETIES
Spring 2022



Homework II
due 11 February

(1) Prove that a polytope $P$ is the affine image of a crosspolytope if and only if $P$ is centrally symmetric.

(2) (**Collaborative**) For today,[1] let's define a *vertex* of a polytope $P$ as a point $\mathbf{v} \in P$ such that for every line $L$ through $\mathbf{v}$ and every neighborhood $N$ of $\mathbf{v}$, there exists a point in $L \cap N$ that is not in $P$. Prove that the vertices of the $n$th Birkhoff–von Neumann polytope are the $n \times n$ permutation matrices.

(3) Let $H_1, H_2, \ldots, H_m$ be halfspaces in $\mathbf{R}^d$ defining the polyhedron $P = \bigcap_{j=1}^m H_j$. This representation of $P$ is *irredundant* if $P \neq \bigcap_{j \in I} H_j$ for any $I \subsetneq [m]$, i.e., we need all $m$ halfspaces to describe $P$. Find a polyhedron $P$ that has two irredundant representations with different $m$'s. (*Hint:* A point in the plane will do.)

(4) The *lineality space* of a polyhedron $P \subseteq \mathbf{R}^d$ is the inclusion-maximal linear subspace $L \subseteq \mathbf{R}^d$ such that $\mathbf{p} + L \subseteq P$ for some $\mathbf{p} \in P$. Now let $P = \{\mathbf{x} \in \mathbf{R}^d : \mathbf{A}\,\mathbf{x} \leq \mathbf{b}\}$. Show that the lineality space of $P$ is the null space of $\mathbf{A}$.

[1] We'll still have a different (more robust) definition of *vertex* soon.