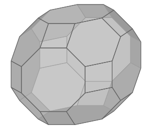
MATH 883: POLYTOPES & VARIETIES
Spring 2022



Homework III
due 18 February

(1) Let $P = \operatorname{conv}(S) \subset \mathbf{R}^d$ be a polytope.
  (a) Let $\mathbf{p}, \mathbf{q} \in S$. Show that if $P = \operatorname{conv}(S \setminus \{\mathbf{p}\}) = \operatorname{conv}(S \setminus \{\mathbf{q}\})$, then $P = \operatorname{conv}(S \setminus \{\mathbf{p}, \mathbf{q}\})$.
  (b) Conclude there is a unique inclusion-minimal set $V \subseteq S$ such that $P = \operatorname{conv}(V)$.

(2) (**Collaborative**) Let C be the V-cone generated by $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$.
  (a) Show that the lineality space of C is nontrivial if and only if there is some $\mathbf{p} \in C \setminus \{\mathbf{0}\}$ such that $-\mathbf{p} \in C$.[1]
  (b) Show that there is some $\mathbf{p} \neq \mathbf{0}$ with $\pm\mathbf{p} \in C$ if and only if there are $\mu_1, \ldots, \mu_k \geq 0$ not all zero such that

$$\mathbf{0} \;=\; \mu_1 \mathbf{u}_1 + \cdots + \mu_k \mathbf{u}_k \,.$$

(3) Given the polyhedron

$$P \;=\; \left\{ \mathbf{x} \in \mathbf{R}^d : \mathbf{a}_i \mathbf{x} \leq b_i \;\text{ for }\; 1 \leq i \leq k \right\},$$

we employ the usual definition of the (topological) *interior* of P: For $\mathbf{p} \in \mathbf{R}^d$ and $\varepsilon > 0$, let $B(\mathbf{p}, \varepsilon)$ be the ball of radius ε centered at $\mathbf{p}$. A point $\mathbf{p} \in P$ is an *interior point* of P if $B(\mathbf{p}, \varepsilon) \subseteq P$ for some $\varepsilon > 0$. Show that the interior of P equals

$$\left\{ \mathbf{x} \in \mathbf{R}^d : \mathbf{a}_i \mathbf{x} < b_i \text{ for } 1 \leq i \leq k \right\}.$$

(It might help to picture, say, the unit square in $\mathbf{R}^2$.)

(4) Given the polyhedron

$$P \;=\; \left\{ \mathbf{x} \in \mathbf{R}^d : \mathbf{a}_i \mathbf{x} \leq b_i \;\text{ for }\; 1 \leq i \leq k \right\},$$

let $I := \{ i \in [k] : \mathbf{a}_i \mathbf{p} = b_i \text{ for all } \mathbf{p} \in P \}$.
  (a) Show that the affine hull $\operatorname{aff}(P)$ of P equals $\{ \mathbf{x} \in \mathbf{R}^d : \mathbf{a}_i \mathbf{x} = b_i \text{ for all } i \in I \}$.
  (b) The *relative interior of* P consists of all points $\mathbf{p}$ with

$$B(\mathbf{p}, \varepsilon) \cap \operatorname{aff}(P) \;\subseteq\; P$$

for some $\varepsilon > 0$. Show that the relative interior of P equals $\{ \mathbf{x} \in P : \mathbf{a}_i \mathbf{x} < b_i \text{ for all } i \notin I \}$.
(Again it might be helpful to consider a running example, such as the square in $\mathbf{R}^3$ with vertices $(1,0,0)$, $(1,1,0)$, $(0,0,1)$, and $(0,1,1)$.)

[1] A cone with trivial lineality space is called *pointed*; so this exercise yields two alternative definitions.