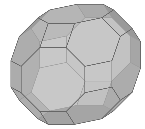
MATH 883: POLYTOPES & VARIETIES
Spring 2022



Homework IV
due 25 February

(1) Describe the cone generated by the vectors $(1,0,0,1)$, $(0,2,3,4)$, $(0,0,3,1)$, $(1,3,2,4)$, $(2,4,6,4)$ by its irredundant linear inequalities.

(2) (**Collaborative**) Recall that the k-permutahedron P is defined as the convex hull of all permutations of $(1,2,\ldots,k)$.
  (a) Compute the affine hull of P.
  (b) Verify that for each $\sigma \in \{0,1\}^k$ we have $\sigma\,\mathbf{x} \geq \frac{|\sigma|(|\sigma|+1)}{2}$ for all $\mathbf{x} \in P$; here $|\sigma|$ denotes the number of 1's in σ. Give an indication that these inequalities and the affine hull define P.
  (c) Prove that each permutation of $(1,2,\ldots,k)$ is indeed a vertex of P.

(3) Let $P \subset \mathbf{R}^d$ and $Q \subset \mathbf{R}^e$ be two polyhedra.
  (a) Show that $P \times Q$ is a polyhedron.
  (b) Now assume $d = e$ and that the relative interiors of P and Q have a nonempty intersection. In this case,[1] we define the *free sum* of P and Q as

$$P \oplus Q \ := \ \mathrm{conv}(P \cup Q)\,.$$

  Show that $P \oplus Q$ is a polyhedron.

(4) Let $P = \mathrm{conv}(\mathbf{p}_1,\ldots,\mathbf{p}_m)$ be a polytope. Show that a point $\mathbf{q}$ is in the relative interior of P if there are $\lambda_1,\ldots,\lambda_m > 0$ such that

$$\mathbf{q} = \lambda_1\mathbf{p}_1 + \cdots + \lambda_m\mathbf{p}_m \quad \text{and} \quad \lambda_1 + \cdots + \lambda_m = 1.$$

When is this an *if and only if* condition?

[1] We can define the free sum of two polyhedra also without this restriction—in fact, for this exercise, the restrictions should not matter—but it won't have nearly as nice properties as under the restriction.