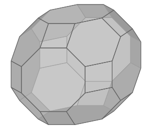
MATH 883: POLYTOPES & VARIETIES
Spring 2022



Homework VI
due 11 March

(1) Compute the face numbers of
  (a) the d-cube
  (b) the d-cross polytope
and verify that they satisfy the Euler–Poincaré relation.

(2) Let P be a polytope with vertex set V, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. Prove that the face $F \in \Phi(P)$ is the join of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if and only if $\frac{1}{k}(\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k) \in F^\circ$.

(3) Fix a polytope P with the origin in its interior. For a nonempty face F of P, let

$$C(F) \;:=\; \bigcup_{t>0} tF^\circ$$

and $C(\varnothing) := \{\mathbf{0}\}$.
  (a) Show that, for each proper face F, the set $C(F)$ is a relatively open cone of dimension $\dim(F)+1$.
  (b) Prove that

$$C(P) \;=\; \operatorname{aff}(P) \;=\; \biguplus_{F \prec P} C(F)\,.$$

(4) For a polyhedron $P \subseteq \mathbf{R}^d$ with lineality space L, we write F/L for the projection of a face F of P in $\mathbf{R}^d/L$.[1]
  (a) Show that F/L is a face of P/L.
  (b) Prove that the map $\Phi(P) \to \Phi(P/L)$ given by $F \mapsto F/L$ is an isomorphism of face lattices.[2]

[1] If you are not a fan of quotient spaces, we can identify F/L with the orthogonal projection of F onto the orthogonal complement of L.

[2] Two posets Φ and Ψ are *isomorphic* if there is a bijection $\Phi \to \Psi$ that respects the order relations in Φ and Ψ.