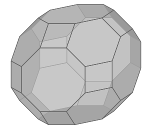
MATH 883: POLYTOPES & VARIETIES
Spring 2022



Homework VII
due 18 March

(1) Show that the function $\chi_{\mathscr{H}} : \mathrm{PC}(\mathscr{H}) \to \mathbf{Z}$ which we defined in class is a valuation.

(2) Let $\Delta = \mathrm{conv}(V)$ be a simplex.
  (a) Prove that $\mathrm{conv}(W)$ is a face of Δ, for any subset $W \subseteq V$, and conclude that the face lattice of Δ is a Boolean lattice.
  (b) Show that, for $-1 \le j \le k \le d$, the number of j-faces contained in any given k-face, equals $\binom{k+1}{j+1}$.

(3) Let $P \subseteq \mathbf{R}^d$ be a polyhedron. For $\mathbf{w} \in \mathbf{R}^d$, let $F_{\mathbf{w}}(P)$ be the face of P that maximizes the linear functional $\mathbf{w}\,\mathbf{x}$, i.e.,

$$F_{\mathbf{w}}(P) \;=\; \{\mathbf{y} \in P : \, \mathbf{w}\,\mathbf{y} \ge \mathbf{w}\,\mathbf{x} \text{ for all } \mathbf{x} \in P\}\,.$$

  (a) Prove that $F_{\mathbf{w}}(P+Q) = F_{\mathbf{w}}(P) + F_{\mathbf{w}}(Q)$.
  (b) Thinking of $[0,1]^d$ as the Minkowski sum of d unit line segments, use (a) to recompute the face lattice of the d-cube.

(4) Recall the definition of a *simplicial* polytope P: all facets are simplices—in poset language, any interval $[\varnothing, F]$ for a facet F of P is a Boolean lattice. Dually, we define a polytope P to be *simple* if any interval $[\mathbf{v}, P]$ for a vertex $\mathbf{v}$ of P is a Boolean lattice.[1] Prove the *Dehn–Sommerville relations*:

$$f_k \;:=\; \sum_{j=0}^{k} (-1)^j \binom{d-j}{d-k} f_j\,.$$

(*Hint:* start with $f_k = \sum_{\substack{F \preceq P \\ \dim F = k}} 1$ and then use the Euler–Poincaré relation for each F.)

[1] There is a nicer definition of a *simple* d-polytope: every vertex is contained in exactly d edges. The easiest way to see how this definition implies that any interval $[\mathbf{v}, P]$ for a vertex $\mathbf{v}$ of P is a Boolean lattice involves the notion of polar duality.