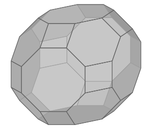
MATH 883: POLYTOPES & VARIETIES
Spring 2022



Homework VIII
due 1 April (not a joke)

(1) Let $\Delta = \mathrm{conv}(\mathbf{u}_0, \ldots, \mathbf{u}_d) \subset \mathbf{R}^d$ and $\Delta' = \mathrm{conv}(\mathbf{v}_0, \ldots, \mathbf{v}_e) \subset \mathbf{R}^e$ be two simplices and let $P := \Delta \times \Delta'$ be their Cartesian product.
   (a) We can identify the vertices of P with nodes of the square grid $\{0, \ldots, d\} \times \{0, \ldots, e\}$. A *lattice path* from $(0,0)$ to (d,e) is a path on the grid that uses only unit steps $\rightarrow$ and $\uparrow$. Show that any such path encodes a unique $(d+e)$-simplex contained in P.
   (b) Show that the collection of all such simplices yields a triangulation of P.

(2) Given a permutation $\tau \in S_d$ on d letters, let

$$\Delta_\tau \ := \ \left\{ \mathbf{x} \in \mathbf{R}^d : 0 \le x_{\tau(1)} \le x_{\tau(2)} \le \cdots \le x_{\tau(d)} \le 1 \right\}.$$

Convince yourself that Δ_τ is a simplex, and prove that $\{\Delta_\tau : \tau \in S_d\}$ yields a triangulation of $[0,1]^d$.

(3) Show that any subdivision of a polygon without new vertices is regular.

(4) (The mother of all nonregular triangulations)
   (a) Prove that the triangulation below (of a triangle, with three additional vertices) is not regular.
   (b) Give an example of a nonregular subdivision of a polytope in every dimension ≥ 3.