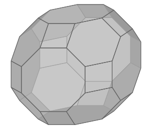
MATH 883: POLYTOPES & VARIETIES
Spring 2022



Homework IX
due 8 April

(1) Let c_n denote the number of triangulations (using only the vertices) of an $(n+2)$-gon.
  (a) Find a recurrence relation for c_n.
  (b) Compute the generating function $C(x) := 1 + \sum_{n \geq 1} c_n x^n$ and derive a closed form for c_n.[1]

(2) Let $P \subseteq \mathbf{R}^d$ be a polyhedron and $\mathbf{x} \in \mathbf{R}^d$. We say that $\mathbf{y} \in P$ *is visible from* $\mathbf{x}$ if the line segment from $\mathbf{x}$ to $\mathbf{y}$ intersects P only in $\mathbf{y}$. Prove that $\mathbf{x} \in \mathbf{R}^d$ is beyond a given face F of P if and only if all points in F are visible from $\mathbf{x}$.

(3) Compute $\mathbb{H}_{\mathbf{q}} \Delta_\tau$ for the simplices in Exercise VIII.2, with $\mathbf{q} = \frac{1}{d+1}(1,2,\dots,d)$.

(4) Fix linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d \in \mathbf{R}^d$ and consider the simplicial cone

$$C \ := \ \mathbf{R}_{\geq \mathbf{0}} \mathbf{v}_1 + \mathbf{R}_{\geq \mathbf{0}} \mathbf{v}_2 + \cdots + \mathbf{R}_{\geq \mathbf{0}} \mathbf{v}_d \, .$$

Prove that, for

$$\hat{C} \ := \ \mathbf{R}_{\geq \mathbf{0}} \mathbf{v}_1 + \cdots + \mathbf{R}_{\geq \mathbf{0}} \mathbf{v}_{m-1} + \mathbf{R}_{>0} \mathbf{v}_m + \cdots + \mathbf{R}_{>0} \mathbf{v}_d \, ,$$

there exists $\mathbf{q} \in \mathbf{R}^d$ (generic relative to C) such that

$$\hat{C} \ = \ \mathbb{H}_{\mathbf{q}} C \, .$$

Conversely, show that, for every generic $\mathbf{q} \in \mathbf{R}^d$ relative to C, the half-open cone $\mathbb{H}_{\mathbf{q}} C$ is of the form $\hat{C}$ for some reordering of the $\mathbf{v}_j$'s and some m.

[1] Here you will need the *binomial series* $(1+x)^\alpha = \sum_{n \geq 0} \binom{\alpha}{n} x^n$, for $\alpha \in \mathbf{R}$, with $\binom{\alpha}{n} := \frac{1}{n!} \alpha(\alpha-1)\cdots(\alpha-n+1)$. Calculus rocks.