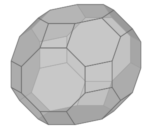
MATH 883: POLYTOPES & VARIETIES
Spring 2022



Homework XI
due 22 April

(1) Compute the Ehrhart polynomial of the octahedron with vertices $(\pm 1,0,0)$, $(0,\pm 1,0)$, and $(0,0,\pm 1)$. Think about how this could generalize to the d-dimensional cross polytope.

(2) Let P be a lattice polytope. Show that the constant term of the h^*-polynomial of P is 1. Conclude that $\mathrm{ehr}_P(0) = 1$.

(3) Let $P \subset \mathbf{R}^d$ be a lattice d-polytope with h^*-polynomial $h_P^*(z) = h_0^* + h_1^* z + \cdots + h_d^* z^d$.[1] Prove:
   (a) $h_1^* = \left|P \cap \mathbf{Z}^d\right| - d - 1$.
   (b) $h_d^* = \left|P^\circ \cap \mathbf{Z}^d\right|$.
   (c) $h_0^* + h_1^* + \cdots + h_d^* = d!\,\mathrm{vol}(P)$.

(4) A *reflexive polytope* is a lattice polytope P such that the origin is the unique interior lattice point of P and

$$\mathrm{ehr}_{P^\circ}(n) \ = \ \mathrm{ehr}_P(n-1) \qquad \text{for all } n \in \mathbf{Z}_{>0}\,.$$

Prove that if P is a lattice d-polytope that contains the origin in its interior and that has the Ehrhart series

$$\mathrm{Ehr}_P(z) \ = \ \frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_1^* z + h_0^*}{(1-z)^{d+1}}\,,$$

then P is reflexive if and only if $h_k^* = h_{d-k}^*$ for all $0 \le k \le \frac{d}{2}$.

[1] $h_P^*(z)$ might not have degree d, i.e., we allow leading coefficients to be 0.