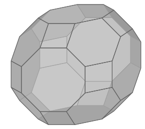
MATH 883: POLYTOPES & VARIETIES
Spring 2022



Homework XII
due in early May

(1) Let P be a polytope. Show that the following statements are equivalent:
  (a) P is a zonotope;
  (b) every 2-dimensional face of P is a zonotope;
  (c) every 2-dimensional face of P is centrally symmetric.;

(2) A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbf{R}^d$ are *in general position* if no $k+1$ of them is contained in a linear subspace of dimension k, for any $1 \leq k \leq d$. Show that all proper faces of a zonotope generated by vectors in general position are parallelepipeds.

(3) Given the zonotope Z generated by the vectors $\pm\mathbf{u}_1, \pm\mathbf{u}_2, \ldots, \pm\mathbf{u}_n \in \mathbf{R}^d$, consider the hyperplane arrangement $\mathscr{H}$ consisting of the n hyperplanes in $\mathbf{R}^d$ through the origin with normal vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Show that there is a one-to-one correspondence between the vertices of Z and the regions of $\mathscr{H}$ (i.e., the maximal connected components of $\mathbf{R}^d \setminus \bigcup \mathscr{H}$). Can you give an analogous correspondence for the other faces of Z?

(4) Let P be the $(d-1)$-dimensional permutahedron (living in $\mathbf{R}^d$). We proved in class that P is a translate of the zonotope Z generated by $\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3, \ldots, \mathbf{e}_{d-1} - \mathbf{e}_d$. Prove that Z can be written as the *disjoint* union

$$Z = \{\mathbf{0}\} \cup \bigcup_{S \in I} \left(\sum_{\mathbf{e}_j - \mathbf{e}_k \in S} (\mathbf{0}, \mathbf{e}_j - \mathbf{e}_k] \right)$$

where I consist of all nonempty linearly independent subsets of $\{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3, \ldots, \mathbf{e}_{d-1} - \mathbf{e}_d\}$.