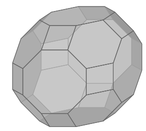
MATH 883: POLYTOPES & VARIETIES
Spring 2022



Homework −I
due 13 May

(1) Suppose $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbf{R}^d$, and randomly choose real numbers $h_1, h_2, \dots, h_m$. Prove that the zonotope generated by $(\mathbf{v}_1, h_1), (\mathbf{v}_2, h_2), \dots, (\mathbf{v}_m, h_m) \in \mathbf{R}^{d+1}$ is *cubical*, i.e., all of its proper faces are parallelepipeds.

(2) Suppose $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbf{R}^d$, randomly choose real numbers $h_1, h_2, \dots, h_m$, and construct the zonotopes Z generated by $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbf{R}^d$ and $\widehat{Z}$ by $(\mathbf{v}_1, h_1), (\mathbf{v}_2, h_2), \dots, (\mathbf{v}_m, h_m) \in \mathbf{R}^{d+1}$.
  (a) Revisit Problems VII(3) and (1) above to show that each bounded face of the polyhedron[1] $\widehat{Z} + \uparrow_{\mathbf{R}}$ is a parallelepiped.
  (b) Let $\pi : \mathbf{R}^{d+1} \to \mathbf{R}^d$ be the projection that forgets the last coordinate. Prove that

$$\left\{ \pi(F) : F \text{ bounded face of } \widehat{Z} + \uparrow_{\mathbf{R}} \right\}$$

is a subdivision of Z into parallelepipeds.[2]

(3) We could take the subdivision in Problem (2) one step further and make it disjoint; we have seen this in one instant, namely Problem XII(4), and so we will concentrate on the special case of permutahedra. In preparation, show that, if $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbf{Z}^d$ are linearly independent, then the Ehrhart polynomial of the half-open parallelelepiped

$$P \ := \ (0,1]\,\mathbf{v}_1 + (0,1]\,\mathbf{v}_2 + \cdots + (0,1]\,\mathbf{v}_m$$

is $\operatorname{ehr}_P(n) = \operatorname{vol}(P)\, n^m$ where $\operatorname{vol}(P)$ denotes relative volume.

(4) Let P be the $(d-1)$-dimensional permutahedron (living in $\mathbf{R}^d$). Recall that in Problem XII(4) you proved that P is a (lattice) translate of

$$\{\mathbf{0}\} \cup \bigcup_{S \in I} \left(\sum_{\mathbf{e}_j - \mathbf{e}_k \in S} \left(\mathbf{0}, \mathbf{e}_j - \mathbf{e}_k\right] \right) \qquad (\star)$$

where I consist of all nonempty linearly independent subsets of $\{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_3, \dots, \mathbf{e}_{d-1} - \mathbf{e}_d\}$.
  (a) Show that each of the parallelepipeds in $(\star)$ has relative volume 1.
  (b) To each subset S in $(\star)$ we associate the graph G_S with nodes $[d]$ and edges $\{jk : \mathbf{e}_j - \mathbf{e}_k \in S\}$. Prove that S is linearly independent if and only if G_S is a forest, i.e., does not contain any cycles.
  (c) Show that the coefficient c_k of the Ehrhart polynomial

$$L_P(t) = c_{d-1}\, t^{d-1} + c_{d-2}\, t^{d-2} + \cdots + c_0$$

of the permutahedron equals the number of labeled forests on d nodes with k edges.

[1] Shall we call it a *zonohedron*?

[2] Such a subdivision is called a *regular tiling* of Z.