The volume of the 10^{th} Birkhoff polytope

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The n^{th} *Birkhoff polytope* is defined as

$$\mathcal{B}_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2} : \ x_{jk} \geq 0, \ \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\} ,$$

often described as the set of all $n \times n$ *doubly stochastic matrices.* $\mathcal{B}_n$ is a convex polytope with integer vertices. A long-standing open problem is the determination of the relative volume of $\mathcal{B}_n$. In [1] we introduced a method of calculating this volume and used it to compute $\operatorname{vol} \mathcal{B}_9$. This note is an update on our progress: with the same program, we have now computed

$\operatorname{vol} \mathcal{B}_{10} =$

$$\frac{727291284016786420977508457990121862548823260052557333386607889}{828160860106766855125676318796872729344622463533089422677980721388055739956270293750883504892820848640000000} .$$

We computed this using the spare time on most of the 50 Linux workstations in the Mathematics department at Binghamton. The total computation time, scaled to a 1GHz processor, was 6160 days, or almost 17 years.

As in [1], we computed part of the *Ehrhart polynomial* of $\mathcal{B}_{10}$, that is, the counting function

$$\# \left(t\mathcal{B}_{10} \cap \mathbb{Z}^{100} \right) \ ,$$

a polynomial in the integer variable t. The leading term of this polynomial is $\operatorname{vol} \mathcal{B}_{10}/10^9$. For details, as well as the computational tricks which were again used in our computation, we refer to our paper [1] and the accompanying web site `www.math.binghamton.edu/dennis/Birkhoff`.

Some final remarks:

1. Unless the editors of *Discrete & Computational Geometry* will allow us to insert these new results in the final version of [1], this note will solely be published on the Mathematics ArXiv (`front.math.ucdavis.edu`).

2. We will *not* attempt to compute $\operatorname{vol} \mathcal{B}_{11}$ with our current algorithm.

References

[1] Matthias Beck and Dennis Pixton, *The Ehrhart polynomial of the Birkhoff polytope*, Preprint (`arXiv:math.CO/0202267`), to appear in Discrete Comput. Geom. (2003).

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