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Arnimallee 2

Discrete Geometry I
Wintersemester 2020/21

Take Home Final Exam   9–11 March 2021

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. You may refer to theorems in the text books. Each of the problems will be weighted equally.

You are welcome to use books and internet sources, but you are not allowed to discuss this exam with anyone (this includes live discussions, calls, chats, etc.). I reserve the right for an oral follow-up meeting if I suspect that you did not follow these rules.

The exam is due on at 12 noon on 11 March 2021 (FU Whiteboard or email), and your submission should be a pdf file (typed or carefully scanned; if you do not have easy access to a scanner, send me a photo by the deadline and a detailed scan by 6pm on 11 March 2021). Please include this page with your name and Matrikelnummer added below, as an indication that you understand and agree with the rules for this exam.

Name ______________________   Matrikelnummer ______________________

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. You may refer to theorems in the text books. You may use software for computations; in this case, tell me what are you computing and underline mathematically that it gives you what you're trying to prove. Each of the problems will be weighted equally.

1. Denote the standard unit vectors in $\mathbb{R}^d$ by $e_1, e_2, ..., e_d$. Let P be the d-dimensional cross polytope, i.e., the convex hull of $\pm e_1, \pm e_2, \ldots, \pm e_d$.

   (a) Compute the signed circuits and corcircuits of the vector configuration associated with P.

   (b) Compute a (linear or affine) Gale diagram of P when $d = 5$.

2. Now let P be the 3-dimensional cross polytope (an octahedron). For each facet F of P, let $S(F)$ be the convex hull of F and the origin. Prove that the set of all such $S(F)$ and their faces form a regular triangulation of P.

3. Let P be a polytope with 0 in its interior. Show that the face fan of P is the normal fan of the polar P^Δ, and the normal fan of P is the face fan of P^Δ.

4. Consider the hyperplane arrangement H in $\mathbb{R}^d$ defined by the equations

$$x_1 = x_2,\ x_2 = x_3,\ \ldots,\ x_{d-1} = x_d,\ x_d = x_1\,.$$

   (a) Compute the vertices of the zonotope defined by H.

   (b) Compute the characteristic polynomial of H and use it to confirm the number of vertices in (a).