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Discrete Geometry II
Sommersemester 2021

**Take Home Final Exam 20–22 July 2021**

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. You may refer to theorems in the text book. Each of the five problems will be weighted equally.

You are welcome to use books and internet sources, but you are not allowed to discuss this exam with anyone (this includes live discussions, calls, chats, etc.). I reserve the right for an oral follow-up exam if I suspect that you did not follow these rules.

The exam is due on at 12 noon on 22 July 2021 (FU Whiteboard or email), and your submission should be a pdf file (typed or carefully scanned). Please include this page with your name and Matrikelnummer added below, as an indication that you understand and agree with the rules for this exam.

Name ______________________ Matrikelnummer ______________________

1. Compute the Ehrhart polynomial of the pyramid with vertices $(0,0,0)$, $(2,0,0)$, $(0,2,0)$, $(2,2,0)$, and $(1,1,1)$.

2. Suppose $\mathcal{P} \subset \mathbb{R}^d$ is a lattice $d$-polytope with Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_s^* \, z^s + h_{s-1}^* \, z^{s-1} + \cdots + h_1^* \, z + 1}{(1-z)^{d+1}} \, .$$

with $h_s^* \neq 0$. Prove that $k\mathcal{P}$ has no interior lattice points for $k = 1, \ldots, d-s$, and that $h_s^* = \#((d-s+1)\mathcal{P}^\circ \cap \mathbb{Z}^d)$.

3. Suppose $f$ and $g$ are quasipolynomials. Prove that their convolution

$$F(t) := \sum_{s=0}^{t} f(s) \, g(t-s)$$

is also a quasipolynomial. What can you say about the degree and the period of $F$, given the degrees and periods of $f$ and $g$? (*Hint:* Use generating functions.)

4. Let $\Delta$ be the tetrahedron with vertices $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, and $(0,0,1)$, where $a$ and $b$ are relatively prime positive integers. Compute the Ehrhart polynomial of $\Delta$. (*Hint:* Use the reciprocity theorem for Dedekind sums.)

5. Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice $d$-polytope that contains the origin in its interior. Recall that we proved that the $h^*$-polynomial of $\mathcal{P}$ has a unique decomposition

$$h^*(z) \;=\; a(z) + z \, b(z)$$

where $a(z)$ and $b(z)$ are palindromic polynomials with $z^d \, a(\frac{1}{z}) = a(z)$ and $z^{d-1} \, b(\frac{1}{z}) = b(z)$. Prove that

$$1 + \sum_{t \geq 1} \left| \partial(t\mathcal{P}) \cap \mathbb{Z}^d \right| z^t \;=\; \frac{a(z)}{(1-z)^d}$$

and

$$\left| \mathcal{P}^\circ \cap \mathbb{Z}^d \right| - 1 + \sum_{t \geq 1} \left| (t+1)\mathcal{P}^\circ \setminus t\mathcal{P} \cap \mathbb{Z}^d \right| z^t \;=\; \frac{b(z)}{(1-z)^d} \, .$$