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DISCRETE GEOMETRY III
Wintersemester 2020/21

Take Home Final Exam   9-11 March 2021

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. You may refer to theorems in the text book. Each of the problems will be weighted equally.

You are welcome to use books and internet sources, but you are not allowed to discuss this exam with anyone (this includes live discussions, calls, chats, etc.). I reserve the right for an oral follow-up exam if I suspect that you did not follow these rules.

The exam is due on at 12 noon on 11 March 2021 (FU Whiteboard or email), and your submission should be a pdf file (typed or carefully scanned). Please include this page with your name and Matrikelnummer added below, as an indication that you understand and agree with the rules for this exam.

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Name   Matrikelnummer

1. Let P be the d-dimensional cross polytope, i.e., the convex hull of the standard unit vectors and their negatives. For each facet F of P, let $S(F)$ be the convex hull of F and the origin. Prove that the set of all such $S(F)$ and their faces form a regular triangulation of P.

2. Compute the characteristic polynomial of the n-dimensional hyperplane arrangements with equations

$$x_1 = x_2, \; x_2 = x_3, \; x_3 = x_4, \; \ldots, \; x_{n-1} = x_n, \; x_n = x_1$$

   and deduce its number of regions.

3. For a half-open simplex $\mathbb{H}_{\mathbf{q}}\Delta$ of dimension r, let

$$m \;:=\; \min \{i : \; h_i^*(\mathbb{H}_{\mathbf{q}}\Delta) \neq 0\}$$

   and

$$m^\circ \;:=\; \max \{i : \; h_i^*(\mathbb{H}_{\mathbf{q}}\Delta) \neq 0\} \, .$$

   Then m and m° are the smallest dilation factors such that $m \, \mathbb{H}_{\mathbf{q}}\Delta$ and $(r+1-m^\circ)\mathbb{H}^{\mathbf{q}}\Delta$ contains lattice points, respectively. (Note that if $\mathbb{H}_{\mathbf{q}}\Delta = \Delta$, then $m = 0$.)

4. Show that an open unimodular d-simplex has f^*-vector $(0, \ldots, 0, 1)$ and deduce from this the f^*-vector of a closed unimodular d-simplex.