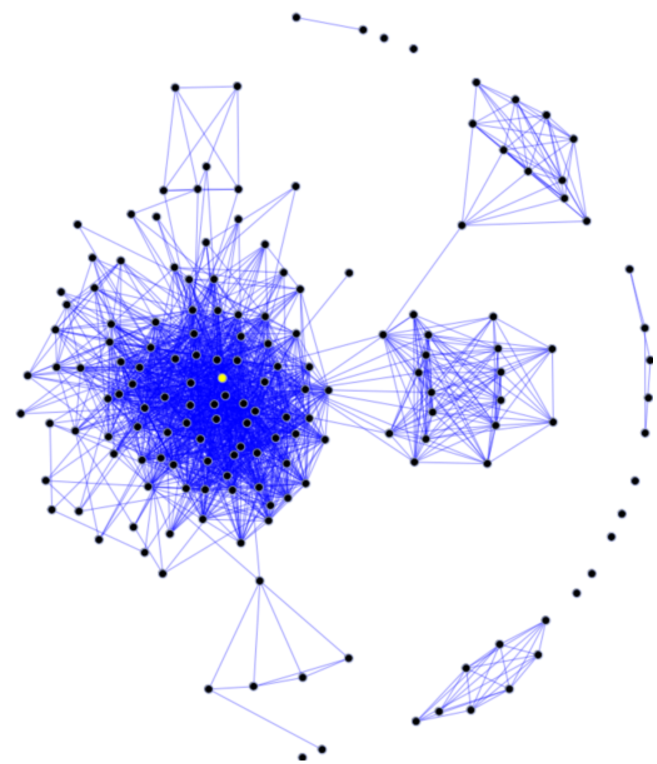
Partially Magic Labelings and the Antimagic Graph Conjecture

Matthias Beck
San Francisco State University
math.sfsu.edu/beck

Maryam Farahmand
UC Berkeley

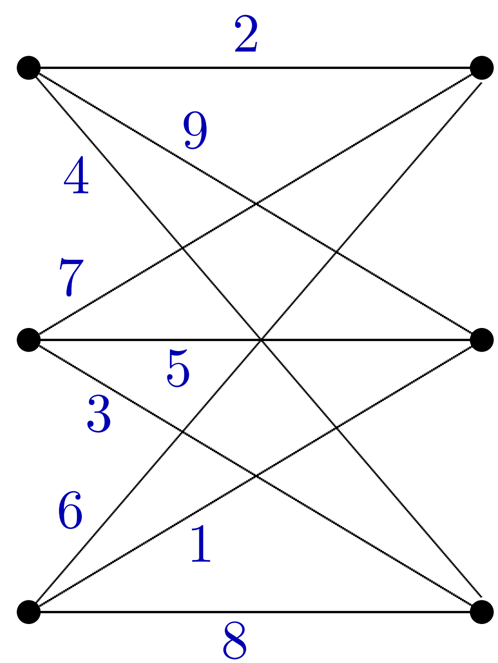
arXiv:1511.04154



[Wikimedia Commons]

Magic Squares & Graphs

2	9	4
7	5	3
6	1	8



A magic labeling of G is an assignment of positive integers to the edges of G such that

- each edge label $1, 2, \ldots, |E|$ is used exactly once;
- the sums of the labels on all edges incident with a given node are equal.

Antimagic Graphs

An antimagic labeling of G is an assignment of positive integers to the edges of G such that

- each edge label $1, 2, \ldots, |E|$ is used exactly once;
- the sum of the labels on all edges incident with a given node is unique.

Antimagic Graphs

An antimagic labeling of G is an assignment of positive integers to the edges of G such that

- each edge label $1, 2, \ldots, |E|$ is used exactly once;
- the sum of the labels on all edges incident with a given node is unique.

Conjecture [Hartsfield & Ringel 1990] Every connected graph except K_2 has an antimagic labeling.

- [Alon et al 2004] connected graphs with minimum degree $\geq c \log |V|$
- [Bérczi et al 2017] connected regular graphs
- open for trees

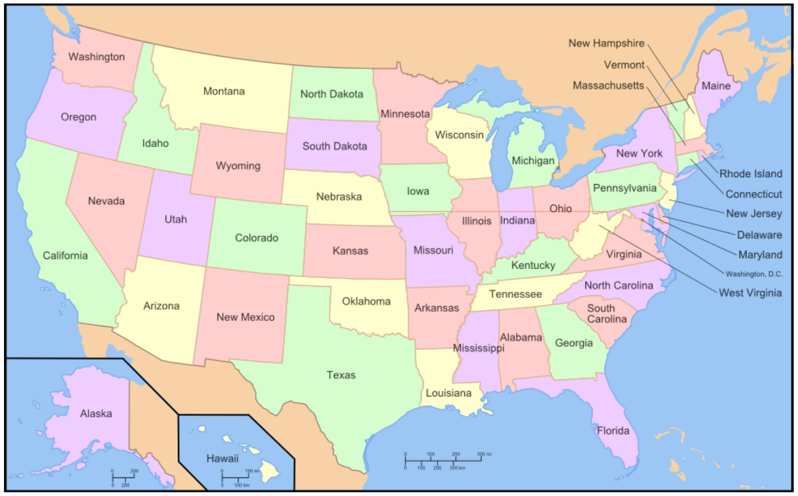
[bart.gov]

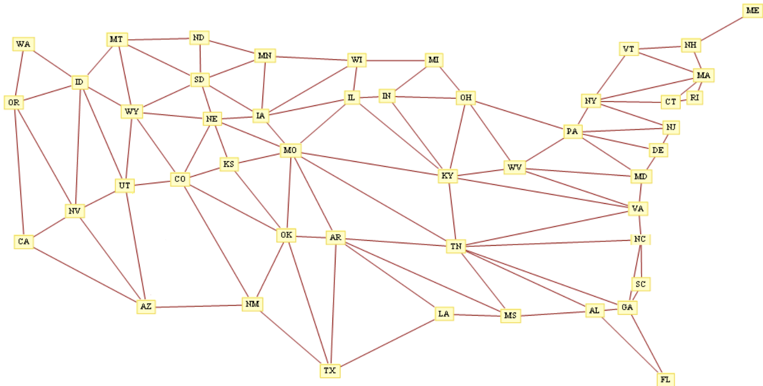
Graph Coloring

Theorem [Appel & Haken 1976]
The chromatic number of any planar graph is at most 4.

This theorem had been a conjecture (conceived by Guthrie when trying to color maps) for 124 years.

Birkhoff [1912] says:
Try polynomials!





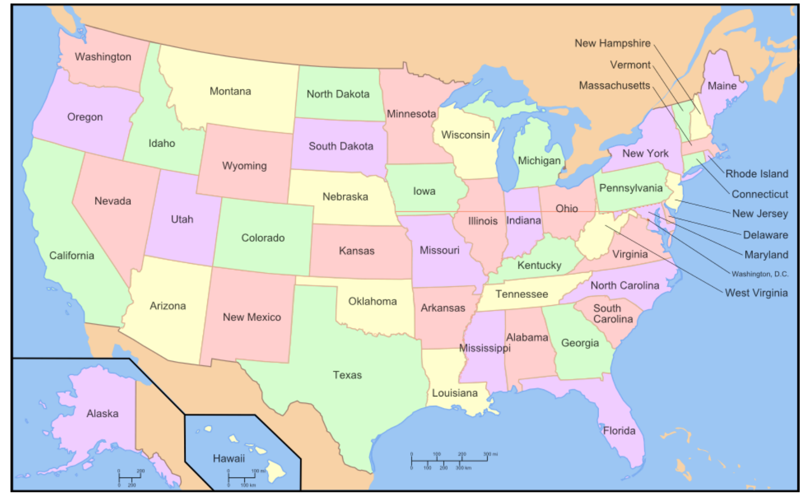
[mathforum.org]

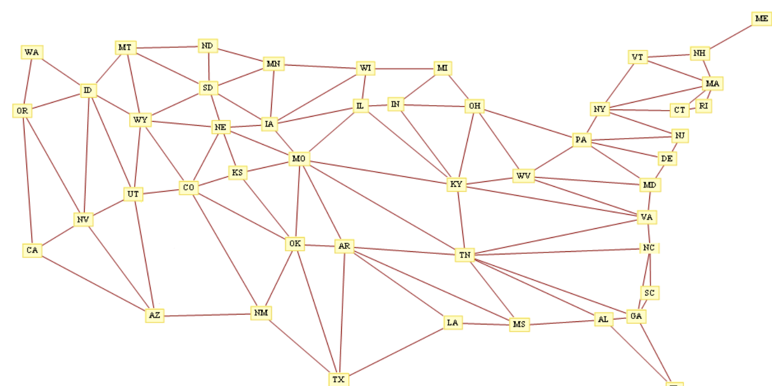
Graph Coloring

Theorem [Appel & Haken 1976]
The chromatic number of any planar graph is at most 4.

This theorem had been a conjecture (conceived by Guthrie when trying to color maps) for 124 years.

Birkhoff [1912] says:
Try polynomials!





[mathforum.org]

Four-Color Theorem Rephrased For a planar graph G, we have $\chi_G(4) > 0$, that is, 4 is not a root of the polynomial $\chi_G(k)$.

Antimagic Counting

An antimagic labeling of G is an assignment of positive integers to the edges of G such that

- each edge label $1, 2, \ldots, |E|$ is used exactly once;
- the sum of the labels on all edges incident with a given node is unique.

Idea Introduce a counting function: let $A_G^*(k)$ be the number of assignments of positive integers to the edges of G such that

- each edge label is in $\{1, 2, \ldots, k\}$ and is distinct;
- the sum of the labels on all edges incident with a given node is unique.

Then G has an antimagic labeling if and only if $A_G^*(|E|) > 0$.

Antimagic Counting

An antimagic labeling of G is an assignment of positive integers to the edges of G such that

- each edge label $1, 2, \ldots, |E|$ is used exactly once;
- the sum of the labels on all edges incident with a given node is unique.

Idea Introduce a counting function: let $A_G^*(k)$ be the number of assignments of positive integers to the edges of G such that

- each edge label is in $\{1, 2, \ldots, k\}$ and is distinct;
- the sum of the labels on all edges incident with a given node is unique.

Then G has an antimagic labeling if and only if $A_G^*(|E|) > 0$.

Bad News The counting function $A_G^*(k)$ is in general not a polynomial:

$$A_{C_4}^*(k) = k^4 - \frac{22}{3}k^3 + 17k^2 - \frac{38}{3}k + \begin{cases} 0 & \text{if } k \text{ is even,} \\ 2 & \text{if } k \text{ is odd.} \end{cases}$$

Antimagic Counting

An antimagic labeling of G is an assignment of positive integers to the edges of G such that

- each edge label $1, 2, \ldots, |E|$ is used exactly once;
- the sum of the labels on all edges incident with a given node is unique.

New Idea Introduce another counting function: let $A_G(k)$ be the number of assignments of positive integers to the edges of G such that

- each edge label is in $\{1, 2, \ldots, k\}$;
- the sum of the labels on all edges incident with a given node is unique.

Antimagic Counting

An antimagic labeling of G is an assignment of positive integers to the edges of G such that

- each edge label $1, 2, \ldots, |E|$ is used exactly once;
- the sum of the labels on all edges incident with a given node is unique.

New Idea Introduce another counting function: let $A_G(k)$ be the number of assignments of positive integers to the edges of G such that

- each edge label is in $\{1, 2, \ldots, k\}$;
- the sum of the labels on all edges incident with a given node is unique.

Theorem (MB–Farahmand) $A_G(k)$ is a quasipolynomial in k of period at most 2. If G minus its loops is bipartite then $A_G(k)$ is a polynomial.

Partially Magic Labelings

A partially magic k-labeling of G over $S \subseteq V$ is an assignment of positive integers to the edges of G such that

- each edge label is in $\{1, 2, \ldots, k\}$;
- the sums of the labels on all edges incident with a node in S are equal.

Let $M_S(k)$ be the number of partially magic k-labeling of G over S.

Partially Magic Labelings

A partially magic k-labeling of G over $S \subseteq V$ is an assignment of positive integers to the edges of G such that

- each edge label is in $\{1, 2, \ldots, k\}$;
- the sums of the labels on all edges incident with a node in S are equal.

Let $M_S(k)$ be the number of partially magic k-labeling of G over S.

Motivation $A_G(k) = \sum_{\substack{S \subseteq V \\ |S| \geq 2}} c_S \, M_S(k)$ for some integers c_S.

Partially Magic Labelings

A partially magic k-labeling of G over $S \subseteq V$ is an assignment of positive integers to the edges of G such that

- each edge label is in $\{1, 2, \ldots, k\}$;
- the sums of the labels on all edges incident with a node in S are equal.

Let $M_S(k)$ be the number of partially magic k-labeling of G over S.

Motivation $A_G(k) = \sum_{\substack{S \subseteq V \\ |S| \geq 2}} c_S \, M_S(k)$ for some integers c_S.

Theorem (MB–Farahmand) $M_S(k)$ is a quasipolynomial in k of period at most 2. If G minus its loops is bipartite then $M_S(k)$ is a polynomial.

For $S = V$ this theorem is due to Stanley [1973].

Two Open Problems

- Directed Antimagic Graph Conjecture [Hefetz–Mütze–Schwartz 2010]
- Distinct Antimagic Counting