Asymptotics of Ehrhart Series of Lattice Polytopes

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Warm-Up Trivia

- Let's say we add two random 100-digit integers. How often should we expect to carry a digit?

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The Eulerian polynomial $A_d(t)$ is defined through $\displaystyle\sum_{m \geq 0} m^d \, t^m = \frac{A_d(t)}{(1-t)^{d+1}}$

Persi Diaconis will tell you that the coefficients of $A_d(t)$ (the Eulerian numbers) play a role here. . .

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Write the Ehrhart h-vector of $\mathcal{P}$ as $h(t) = h_d t^d + h_{d-1} t^{d-1} + \cdots + h_0$, then

$$L_{\mathcal{P}}(m) = \sum_{j=0}^{d} h_j \binom{m+d-j}{d} .$$

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Easier Problem Study $\mathrm{Ehr}_{n\mathcal{P}}(t) = 1 + \sum_{m \ge 1} L_{\mathcal{P}}(nm)\, t^m$ as n increases.

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- Polytopes are basic geometric objects, yet even for these basic objects volume computation is hard and there remain many open problems.
- Many discrete problems in various mathematical areas are linear problems, thus they ask for the discrete volume of a polytope in disguise.
- Polytopes are cool.

General Properties of Ehrhart h-Vectors

$$\mathrm{Ehr}_{\mathcal{P}}(t) = 1 + \sum_{m \ge 1} \# \left(m\mathcal{P} \cap \mathbb{Z}^d \right) t^m = \frac{h_d t^d + h_{d-1} t^{d-1} + \cdots + h_0}{(1-t)^{d+1}}$$

- (Ehrhart) $h_0 = 1$, $h_1 = \# \left(\mathcal{P} \cap \mathbb{Z}^d \right) - d - 1$, $h_d = \# \left(\mathcal{P}^\circ \cap \mathbb{Z}^d \right)$

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- (Stanley 1991) Whenever $h_s > 0$ but $h_{s+1} = \cdots = h_d = 0$, then $h_0 + h_1 + \cdots + h_j \leq h_s + h_{s-1} + \cdots + h_{s-j}$ for all $0 \leq j \leq s$.

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- (Hibi 1994) $h_0 + \cdots + h_{j+1} \geq h_d + \cdots + h_{d-j}$ for $0 \leq j \leq \left\lfloor \frac{d}{2} \right\rfloor - 1$.

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- (Stapledon 2009) Many more inequalities for the h_j's arising from Kneser's Theorem (arXiv:0904.3035)

General Properties of Ehrhart h-Vectors

A triangulation τ of $\mathcal{P}$ is unimodular if for any simplex of τ with vertices $v_0, v_1, \dots, v_d$, the vectors $v_1 - v_0, \dots, v_d - v_0$ form a basis of $\mathbb{Z}^d$.

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- Recent papers of Reiner–Welker and Athanasiadis use this as a starting point to give conditions under which the Ehrhart h-vector is unimodal, i.e., $h_d \le h_{d-1} \le \cdots \le h_k \ge h_{k-1} \ge \cdots \ge h_0$ for some k.

The Main Question

Define $h_0(n), h_1(n), \ldots, h_d(n)$ through

$$\mathrm{Ehr}_{n\mathcal{P}}(t) = \frac{h_d(n)\, t^d + h_{d-1}(n)\, t^{d-1} + \cdots + h_0(n)}{(1-t)^{d+1}} \, .$$

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Let $h(t) = (1-t)^{d+1} \, \mathrm{Ehr}_{\mathcal{P}}(t)$. The operator U_n defined through

$$\mathrm{Ehr}_{n\mathcal{P}}(t) = 1 + \sum_{m \ge 1} L_{\mathcal{P}}(nm)\, t^m = \frac{\mathrm{U}_n \, h(t)}{(1-t)^{d+1}}$$

appears in Number Theory as a Hecke operator and in Commutative Algebra in Veronese subring constructions.

Motivation I: Unimodular Triangulations

Theorem (Kempf–Knudsen–Mumford–Saint-Donat–Waterman 1970's)
For every lattice polytope $\mathcal{P}$ there exists an integer m such that $m\mathcal{P}$ admits a regular unimodular triangulation.

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Theorem (Athanasiadis–Hibi–Stanley 2004) If the d-dimensional lattice polytope $\mathcal{P}$ admits a regular unimodular triangulation, then the Ehrhart h-vector of $\mathcal{P}$ satisfies

(a) $h_{j+1} \geq h_{d-j}$ for $0 \leq j \leq \lfloor \frac{d}{2} \rfloor - 1$,

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There are (many) lattice polytopes for which (some of these) inequalities fail and one may hope to use this theorem to construct a counter-example to the Knudsen–Mumford–Waterman Conjectures.

Veronese Polynomials Are Eventually Unimodal

Theorem (Brenti–Welker 2008) For any $d \in \mathbb{Z}_{>0}$, there exists real numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_{d-1} < \alpha_d = 0$, such that, if $h(t)$ is a polynomial of degree at most d with nonnegative integer coefficients and constant term 1, then for n sufficiently large, $\mathrm{U}_n \, h(t)$ has negative real roots $\beta_1(n) < \beta_2(n) < \cdots < \beta_{d-1}(n) < \beta_d(n) < 0$ and $\lim_{n \to \infty} \beta_j(n) = \alpha_j$.

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If the polynomial $p(t) = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_0$ has negative roots, then its coefficients are (strictly) log concave ($a_j^2 > a_{j-1} a_{j+1}$).

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A General Theorem

The Eulerian polynomial $A_d(t)$ is defined through $\displaystyle\sum_{m\ge0} m^d\, t^m = \frac{A_d(t)}{(1-t)^{d+1}}$.

Theorem (MB–Stapledon) Fix a positive integer d and let $\rho_1 < \rho_2 < \cdots < \rho_d = 0$ denote the roots of $A_d(t)$. There exist M, N depending only on d such that, if $h(t)$ is a polynomial of degree at most d with nonnegative integer coefficients and constant term 1, then for $n \ge N$, $\mathrm{U}_n\, h(t)$ has negative real roots $\beta_1(n) < \beta_2(n) < \cdots < \beta_{d-1}(n) < \beta_d(n) < 0$ with $\lim_{n\to\infty} \beta_j(n) = \rho_j$, and the coefficients of $\mathrm{U}_n\, h(t)$ satisfy $h_j(n) < M\, h_d(n)$.

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In particular, the coefficients of $\mathrm{U}_n\, h(t)$ are unimodal for $n \geq N$.

Furthermore, if $h_0 + \cdots + h_{j+1} \geq h_d + \cdots + h_{d-j}$ for $0 \leq j \leq \left\lfloor \frac{d}{2} \right\rfloor - 1$, with at least one strict inequality, then we may choose N such that, for $n \geq N$,

$$h_0 = h_0(n) < h_d(n) < h_1(n) < \cdots < h_j(n) < h_{d-j}(n) < h_{j+1}(n)$$
$$< \cdots < h_{\lfloor \frac{d+1}{2} \rfloor}(n) < M\, h_d(n)\,.$$

An Ehrhartian Corollary

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Stapledon's Decomposition A polynomial $h(t) = h_{d+1}t^{d+1} + h_d t^d + \cdots + h_0$ of degree at most $d+1$ has a unique decomposition $h(t) = a(t) + b(t)$ where $a(t)$ and $b(t)$ are polynomials satisfying $a(t) = t^d\, a(\frac{1}{t})$ and $b(t) = t^{d+1}\, b(\frac{1}{t})$.

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Corollary Hibi's inequalities $h_0 + \cdots + h_{j+1} \geq h_d + \cdots + h_{d-j}$ for the Ehrhart h-vector are strict.

Ingredients II

Let $h(t) = h_{d+1}t^{d+1} + h_d t^d + \cdots + h_0$ be a polynomial of degree at most $d+1$ and expand $\frac{h(t)}{(1-t)^{d+1}} = h_0 + \sum_{m \ge 1} g(m)\, t^m$, for some polynomial $g(m) = g_d m^d + g_{d-1} m^{d-1} + \cdots + g_0$.

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Theorem (Betke–McMullen 1985) If $h_j \geq 0$ for $0 \leq j \leq d+1$, then for any $1 \leq r \leq d-1$,

$$g_r \leq (-1)^{d-r} S_r(d)\, g_d + \frac{(-1)^{d-r-1}\, h_0\, S_{r+1}(d)}{(d-1)!} \,,$$

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Ingredients III

Let $h(t) = h_d m^d + h_{d-1} m^{d-1} + \cdots + h_0$ be a polynomial of degree at most d and expand $\frac{h(t)}{(1-t)^{d+1}} = \sum_{m \geq 0} g(m)\, t^m$, for some polynomial $g(m) = g_d m^d + g_{d-1} m^{d-1} + \cdots + g_0$.

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Recall our notation $\mathrm{U}_n\, h(t) = h_d(n)\, t^d + h_{d-1}(n)\, t^{d-1} + \cdots + h_0(n)$.

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In particular, for $1 \le j \le d$, $h_j(n)$ is a polynomial in n of degree d and

$$h_j(n) = A(d,j)\, g_d\, n^d + (A(d-1,j) - A(d-1,j-1))\, g_{d-1}\, n^{d-1} + O(n^{d-2})\,.$$

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Theorem (Cauchy) Let $p(n) = p_d\, n^d + p_{d-1}\, n^{d-1} + \cdots + p_0$ be a polynomial of degree d with real coefficients. The complex roots of $p(n)$ lie in the open disc

$$\left\{ z \in \mathbb{C} : \ |z| < 1 + \max_{0 \le j \le d} \left| \frac{p_j}{p_d} \right| \right\} .$$

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Let $R = \oplus_{j \geq 0} R_j$ be a graded ring; we assume that $R_0 = K$ is a field and that R is finitely generated over K.

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Example Denote the cone over $\mathcal{P} \times \{1\}$ by $\operatorname{cone} \mathcal{P}$. Then the semigroup algebra $K[\operatorname{cone} \mathcal{P} \cap \mathbb{Z}^{d+1}]$ (graded by the projection to the last coordinate) gives rise to the Hilbert function $H(K[\operatorname{cone} \mathcal{P} \cap \mathbb{Z}^{d+1}], m) = L_{\mathcal{P}}(m)$.

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Corollary (MB–Stapledon) Fix a positive integer d and let $\rho_1 < \rho_2 < \cdots < \rho_d = 0$ denote the roots of the Eulerian polynomial $A_d(t)$. There exist M, N depending only on d such that, if $R = \bigoplus_{j \geq 0} R_j$ is a finitely generated graded ring over a field $R_0 = K$ that is Cohen–Macauley and module finite over the K-subalgebra generated by R_1, and if the Hilbert function $H(R, m)$ is a polynomial in m, then for $n \geq N$, $\mathrm{U}_n \, h(t)$ has negative real roots $\beta_1(n) < \beta_2(n) < \cdots < \beta_{d-1}(n) < \beta_d(n) < 0$ with $\lim_{n \to \infty} \beta_j(n) = \rho_j$.

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Open Problems

Find optimal choices for M and N in any of our theorems.

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Conjecture For Ehrhart series of d-dimensional polytopes, $N = d$.

(Open for $d \geq 3$)

One Result about Explicit Bounds

Recall our inequalities $h_{j+1}(n) > h_{d-j}(n)$ in the main theorem. . .

Theorem (MB–Stapledon) Fix a positive integer d and set $N = d$ if d is even and $N = \frac{d+1}{2}$ if d is odd. If $h(t)$ is a polynomial of degree at most d satisfying $h_0 + \cdots + h_{j+1} > h_d + \cdots + h_{d-j}$ for $0 \le j \le \left\lfloor \frac{d}{2} \right\rfloor - 1$, then the coefficients of $\mathrm{U}_n \, h(t)$ satisfy $h_{j+1}(n) > h_{d-j}(n)$ for $0 \le j \le \left\lfloor \frac{d}{2} \right\rfloor - 1$ and $n \ge N$.

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Corollary Fix a positive integer d and set $N = d$ if d is even and $N = \frac{d+1}{2}$ if d is odd. If P is a d-dimensional lattice polytope with Ehrhart h-vector $h(t)$, then the coefficients of $\mathrm{U}_n \, h(t)$ satisfy $h_{j+1}(n) > h_{d-j}(n)$ for $0 \le j \le \left\lfloor \frac{d}{2} \right\rfloor - 1$ and $n \ge N$.

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Homework Figure out what all of this has to do with carrying digits when summing 100-digit numbers.