Relations for Barnes Zeta Functions

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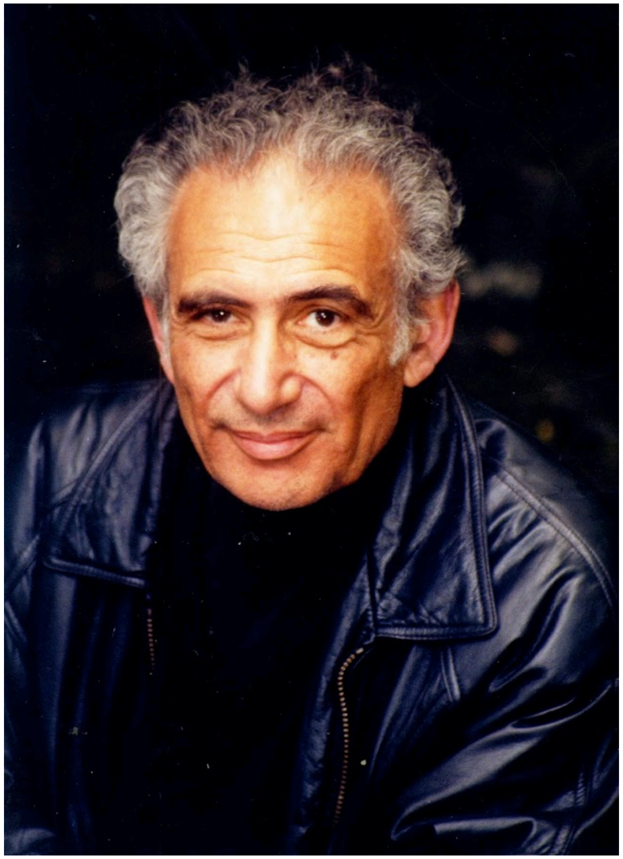

In fond memory of my teacher, mentor, and friend Marvin Knopp

COMBINATORICS/ MODULAR FUNCTIONS/

NUMBER THEORY

SEMINAR

WED, DEC 2, 1998, 2:40 PM, CB617

Ted Chinburg (U Penn) will speak on "Number theory and the color of Gray codes."

WED, DEC 9, 1998, 2^{40} PM, CB617

Matthias Beck will speak on

(NOTE SPELLING) "Counting lattice points by means of the residue theorem".

WED, NOV. 25, 1998, 2^{40} PM, CB617:

NO MEETING

Bernoulli Relations

$$\frac{z}{e^z - 1} = \sum_{k \geq 0} B_k \frac{z^k}{k!} \qquad \xrightarrow[\text{et al}]{\text{Euler}} \qquad \sum_{j=0}^{n} \binom{n}{j} B_j \, B_{n-j} = -n \, B_{n-1} - (n-1) B_n$$

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Nörlund (1922): Relations for Bernoulli polynomials $B_k(x)$ defined through

$$\frac{z \, e^{xz}}{e^z - 1} = \sum_{k \geq 0} B_k(x) \frac{z^k}{k!}$$

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Dilcher (1996): Relations for Bernoulli numbers of order n defined through

$$\left(\frac{z}{e^z - 1} \right)^n = \sum_{k \geq 0} B_k^{(n)} \frac{z^k}{k!}$$

and their polynomial generalization.

Bernoulli Relations

$$\frac{z}{e^z - 1} = \sum_{k \ge 0} B_k \frac{z^k}{k!} \qquad \overset{\text{Euler}}{\underset{\text{et al}}{\longrightarrow}} \qquad \sum_{j=0}^{n} \binom{n}{j} B_j \, B_{n-j} = -n \, B_{n-1} - (n-1) B_n$$

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Goal: Relations for Bernoulli–Barnes numbers $B_k(\mathbf{a})$ defined through

$$\frac{z^n}{(e^{a_1 z} - 1) \cdots (e^{a_n z} - 1)} = \sum_{k \ge 0} B_k(\mathbf{a}) \frac{z^k}{k!} , \qquad \mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n_{>0}$$

Bernoulli–Barnes Relations

$$\frac{z^n}{(e^{a_1 z}-1)\cdots(e^{a_n z}-1)} = \sum_{k\ge 0} B_k(\mathbf{a})\frac{z^k}{k!}\,, \qquad \mathbf{a}=(a_1,a_2,\dots,a_n)\in\mathbb{R}^n_{>0}$$

Theorem 1 For $n \ge 3$ and odd $m \ge 1$

$$\sum_{j=n-m}^{n} \binom{n+j-4}{j-2} \frac{1}{(m-n+j)!} \sum_{|I|=j} B_{m-n+j}(\mathbf{a}_I) = \begin{cases} \frac{1}{2} & \text{if } n=m=3 \\ 0 & \text{otherwise} \end{cases}$$

where the inner sum is over all subsets $I \subseteq \{1,2,\dots,n\}$ of cardinality j and $\mathbf{a}_I := (a_i : i \in I)$.

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where the inner sum is over all subsets $I \subseteq \{1,2,\dots,n\}$ of cardinality j and $\mathbf{a}_I := (a_i : i \in I)$.

Corollary For $n \geq 3$ and odd $m \geq n-2$

$$\sum_{j=2}^{n} \binom{n+j-4}{j-2} \frac{m!}{(m-n+j)!} \binom{n}{j} B^{(j)}_{m-n+j} = \begin{cases} 3 & \text{if } n=m=3 \\ 0 & \text{otherwise} \end{cases}$$

Bernoulli–Barnes Relations

$$\frac{z^n}{(e^{a_1 z}-1)\cdots(e^{a_n z}-1)} = \sum_{k \ge 0} B_k(\mathbf{a}) \frac{z^k}{k!}, \qquad \mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n_{>0}$$

Theorem 1 For $n \ge 3$ and odd $m \ge 1$

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where the inner sum is over all subsets $I \subseteq \{1, 2, \dots, n\}$ of cardinality j, and $\mathbf{a}_I := (a_i : i \in I)$.

Poof Don't use a Siegel-type integration path with integrand

$$\frac{z^{s-1}}{(e^{a_1 z}-1)(e^{a_2 z}-1)\cdots(e^{a_n z}-1)}$$

Bernoulli–Barnes Relations

$$\frac{z^n}{(e^{a_1 z}-1)\cdots(e^{a_n z}-1)} = \sum_{k\ge 0} B_k(\mathbf{a}) \frac{z^k}{k!}\,, \qquad \mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n_{>0}$$

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where the inner sum is over all subsets $I \subseteq \{1,2,\dots,n\}$ of cardinality j, and $\mathbf{a}_I := (a_i : i \in I)$.

Proof idea Show that

$$\sum_{j=2}^{n} \binom{n+j-4}{j-2} (-z)^{n-j} \sum_{|I|=j} \frac{z^{|I|} e^{z \sum_{i\in I} a_i}}{\prod_{i\in I}(e^{a_i z}-1)}$$

is an even function of z.

Barnes Zeta Functions

$$\zeta_n(z,x;\mathbf{a}) := \sum_{\mathbf{m}\in\mathbb{Z}_{\geq 0}^n} \frac{1}{(x+m_1a_1+\cdots+m_na_n)^z}$$

defined for $\mathrm{Re}(x)>0,\ \mathrm{Re}(z)>n$ and continued meromorphically to $\mathbb{C}$.

$$\mathbf{a}=(1,1,\ldots,1) \qquad \longrightarrow \qquad \zeta_n(s;x) := \zeta(s;x,(1,\ldots,1))$$

is the Hurwitz zeta function of order n. The Hurwitz zeta function is the special case $n=1$, the Riemmann zeta function the special case $x=1$.

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$$\zeta_n(-k,x;\mathbf{a}) = \frac{(-1)^n k!}{(k+n)!} B_{k+n}(x;\mathbf{a})$$

where $B_k(x;\mathbf{a})$ is a Bernoulli–Barnes polynomial defined through

$$\frac{z^n e^{xz}}{(e^{a_1z}-1)\cdots(e^{a_nz}-1)} = \sum_{k\ge 0} B_k(x;\mathbf{a})\frac{z^k}{k!}$$

Note that $B_k(\mathbf{a}) = B_k(0;\mathbf{a})$

Barnes Zeta Relations

$$\zeta_n(z,x;\mathbf{a}) := \sum_{\mathbf{m}\in\mathbb{Z}^n_{\geq 0}} \frac{1}{\left(x+m_1a_1+\cdots+m_na_n\right)^z}$$

$$\frac{z^n e^{xz}}{\left(e^{a_1z}-1\right)\cdots\left(e^{a_nz}-1\right)} = \sum_{k\geq 0} B_k(x;\mathbf{a})\frac{z^k}{k!}$$

Theorem 2 Let $a_1,\ldots,a_n$ be pairwise coprime positive integers. Then

$$\begin{aligned}\zeta(s;x,\mathbf{a}) \quad &= \quad \frac{(-1)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}B_{n-1-k}(x;\mathbf{a})\,\zeta(s-k;x)\\ &\quad + \sum_{j=1}^{n} a_j^{-s}\sum_{r=0}^{a_j-1}\sigma_{-r}(a_1,\ldots,\widehat{a}_j,\ldots,a_n;a_j)\,\zeta\left(s;\frac{x+r}{a_j}\right)\end{aligned}$$

where $\sigma_r\left(a_1,\ldots,\widehat{a}_j\ldots,a_n;a_j\right) := \dfrac{1}{a_j}\displaystyle\sum_{m=1}^{a_j-1}\frac{e^{2\pi i m r/a_j}}{\prod_{k\neq j}\left(1-e^{2\pi i m a_k/a_j}\right)}$

is a Fourier–Dedekind sum.

Reciprocity Theorems

Theorem 2 Let $a_1, \ldots, a_n$ be pairwise coprime positive integers. Then

$$\zeta(s;x,\mathbf{a}) = \frac{(-1)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}B_{n-1-k}(x;\mathbf{a})\,\zeta(s-k;x)$$
$$+\sum_{j=1}^{n}a_j^{-s}\sum_{r=0}^{a_j-1}\sigma_{-r}(a_1,\ldots,\widehat{a}_j,\ldots,a_n;a_j)\,\zeta\left(s;\frac{x+r}{a_j}\right).$$

Corollary [$n=2$] Let a, b be coprime positive integers. Then

$$\zeta(s;x,(a,b)) = \frac{1}{ab}\zeta(s-1;x) + \left(1-\frac{x}{ab}\right)\zeta(s;x)$$
$$-a^{-s}\sum_{r=0}^{a-1}\left\{\frac{b^{-1}r}{a}\right\}\zeta\left(s;\frac{x+r}{a}\right) - b^{-s}\sum_{r=0}^{a-1}\left\{\frac{a^{-1}r}{b}\right\}\zeta\left(s;\frac{x+r}{b}\right).$$

Reciprocity Theorems

Corollary [$n = 2$] Let a, b be coprime positive integers. Then

$$\zeta(s; x, (a,b)) = \frac{1}{ab}\zeta(s-1; x) + \left(1 - \frac{x}{ab}\right)\zeta(s; x)$$
$$-a^{-s}\sum_{r=0}^{a-1}\left\{\frac{b^{-1}r}{a}\right\}\zeta\left(s; \frac{x+r}{a}\right) - b^{-s}\sum_{r=0}^{a-1}\left\{\frac{a^{-1}r}{b}\right\}\zeta\left(s; \frac{x+r}{b}\right).$$

Corollary [$s \in \mathbb{Z}_{<0}$] Let a, b be coprime positive integers. Then

$$a^m\sum_{r=0}^{a-1}\left\{\frac{b^{-1}r}{a}\right\}B_{m+1}\left(\frac{x+r}{a}\right) + b^m\sum_{r=0}^{a-1}\left\{\frac{a^{-1}r}{b}\right\}B_{m+1}\left(\frac{x+r}{b}\right) =$$
$$\frac{1}{m+2}B_{m+2}(x, (a,b)) + \frac{1}{ab}\frac{m+1}{m+2}B_{m+2}(x) + \left(\frac{x}{ab} - 1\right)B_{m+1}(x).$$

This is reminiscent of reciprocity theorems for Dedekind sums. . .

Reciprocity Theorems

$$a^m \sum_{r=0}^{a-1} \left\{ \frac{b^{-1} r}{a} \right\} B_{m+1} \left(\frac{x+r}{a} \right) + b^m \sum_{r=0}^{a-1} \left\{ \frac{a^{-1} r}{b} \right\} B_{m+1} \left(\frac{x+r}{b} \right) =$$

$$\frac{1}{m+2} B_{m+2}(x, (a,b)) + \frac{1}{ab} \frac{m+1}{m+2} B_{m+2}(x) + \left(\frac{x}{ab} - 1 \right) B_{m+1}(x)$$

is a polynomial generalization of Apostol's reciprocity law

$$\frac{1}{m} \left(a^{m-1} s_m(a,b) + b^{m-1} s_m(b,a) \right) =$$

$$\sum_{i=0}^{m+1} \binom{m+1}{i} (-1)^{m+1-i} a^i b^{m+1-i} B_i B_{m+1-i}$$

for

$$S_m(a,b) := \sum_{r=0}^{a-1} \left\{ \frac{a^{-1} r}{b} \right\} B_m \left(\frac{r}{b} \right) = \sum_{r=0}^{a-1} \frac{r}{b} B_m \left(\left\{ \frac{ar}{b} \right\} \right) .$$

The case $m = 1$ gives Dedekind sums and their reciprocity law.

Hurwitz Zeta Relations

Theorem 2 Let $a_1, \dots, a_n$ be pairwise coprime positive integers. Then

$$\begin{aligned}\zeta(s;x,\mathbf{a}) \quad &= \quad \frac{(-1)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}B_{n-1-k}(x;\mathbf{a})\,\zeta(s-k;x) \\ &\quad + \sum_{j=1}^{n} a_j^{-s}\sum_{r=0}^{a_j-1}\sigma_{-r}(a_1,\dots,\widehat{a}_j,\dots,a_n;a_j)\,\zeta\left(s;\frac{x+r}{a_j}\right).\end{aligned}$$

Corollary $[\ \mathbf{a}=(1,1,\dots,1)\ ]$

$$\zeta_n(s;x)=\frac{(-1)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}B^{(n)}_{n-1-k}(x)\,\zeta(s-k;x)$$

Hurwitz Zeta Relations

Corollary [$\mathbf{a} = (1, 1, \ldots, 1)$]

$$\zeta_n(s;x) = \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} B^{(n)}_{n-1-k}(x) \, \zeta(s-k;x)$$

Corollary [$s \in \mathbb{Z}_{<0}$] For any positive integers m, n

$$B^{(n)}_{m+n}(x) = (m+n) \binom{m+n-1}{n-1} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} B^{(n)}_{n-1-k}(x) \frac{B_{m+k+1}(x)}{m+k+1}$$

Hurwitz Zeta Relations

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This recovers once more Dilcher's and Euler's relations for Bernoulli numbers and polynomials.

Barnes Zeta Relations

Theorem 2 Let $a_1, \dots, a_n$ be pairwise coprime positive integers. Then

$$\zeta(s; x, \mathbf{a}) = \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} B_{n-1-k}(x; \mathbf{a})\, \zeta(s-k; x) + \sum_{j=1}^{n} a_j^{-s} \sum_{r=0}^{a_j-1} \sigma_{-r}(a_1, \dots, \widehat{a}_j, \dots, a_n; a_j)\, \zeta\left(s; \frac{x+r}{a_j}\right).$$

Proof idea We can write

$$\zeta(s; x, \mathbf{a}) = \sum_{m_1, \dots, m_n \geq 0} \frac{1}{(x + m_1 a_1 + \cdots + m_n a_n)^s} = \sum_{t \geq 0} \frac{p_A(t)}{(x+t)^s}$$

where

$$p_A(t) := \# \left\{ (k_1, \dots, k_n) \in \mathbb{Z}_{\geq 0}^n : \, k_1 a_1 + \cdots + k_n a_n = t \right\}$$

counts all partitions of t with parts in the finite set $A := \{a_1, \dots, a_n\}$.

Barnes Zeta Relations

Theorem 2 Let $a_1, \ldots, a_n$ be pairwise coprime positive integers. Then

$$\zeta(s; x, \mathbf{a}) = \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} B_{n-1-k}(x; \mathbf{a}) \, \zeta(s-k; x)$$
$$+ \sum_{j=1}^{n} a_j^{-s} \sum_{r=0}^{a_j - 1} \sigma_{-r}(a_1, \ldots, \widehat{a}_j, \ldots, a_n; a_j) \, \zeta\left(s; \frac{x+r}{a_j}\right)$$

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$$\zeta(s; x, \mathbf{a}) = \sum_{m_1, \ldots, m_n \geq 0} \frac{1}{(x + m_1 a_1 + \cdots + m_n a_n)^s} = \sum_{t \geq 0} \frac{p_A(t)}{(x+t)^s}$$

and realize that $p_A(t)$ can be expressed using Barnes–Bernoulli polynomials and Fourier–Dedekind sums. . .