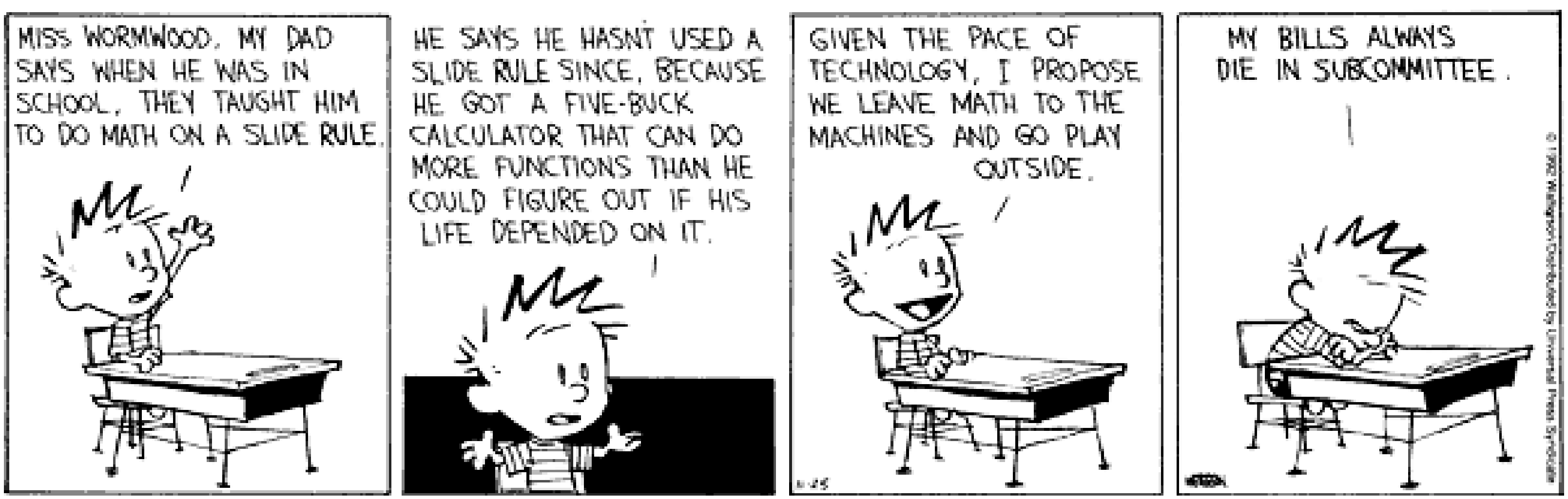
Computing the continuous discretely: The magic quest for a volume

Matthias Beck

San Francisco State University

math.sfsu.edu/beck



Joint work with...

- Dennis Pixton (Birkhoff volume)
- Ricardo Diaz and Sinai Robins (Fourier-Dedekind sums)
- Ira Gessel and Takao Komatsu (restricted partition function)
- Jesus De Loera, Mike Develin, Julian Pfeifle, Richard Stanley (roots of Ehrhart polynomials)

Birkhoff polytope

$$\mathcal{B}_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \in \mathbb{R}_{\geq 0}^{n^2} : \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\}$$

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- $\mathcal{B}_n$ is a convex polytope of dimension $(n-1)^2$
- Vertices are the $n \times n$-permutation matrices.

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One approach: for $X \subset \mathbb{R}^d$, $\operatorname{vol} X = \lim_{t \to \infty} \frac{\#\left(tX \cap \mathbb{Z}^d\right)}{t^d}$

(Weak) semimagic squares

$$\begin{aligned} H_n(t) &:= \# \left(t\mathcal{B}_n \cap \mathbb{Z}^{n^2} \right) \\ &= \# \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \in \mathbb{Z}_{\geq 0}^{n^2} : \begin{array}{l} \sum_j x_{jk} = t \\ \sum_k x_{jk} = t \end{array} \right\} \end{aligned}$$

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Theorem (Ehrhart, Stanley 1973, conjectured by Anand-Dumir-Gupta 1966) $H_n(t)$ is a polynomial in t of degree $(n-1)^2$.

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For example...

- $H_1(t) = 1$
- $H_2(t) = t + 1$
- (MacMahon 1905) $H_3(t) = 3\binom{t+3}{4} + \binom{t+2}{2} = \frac{1}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 + \frac{9}{4}t + 1$

Ehrhart quasi-polynomials

Rational (convex) polytope $\mathcal{P}$ – convex hull of finitely many points in $\mathbb{Q}^d$

Alternative description: $\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \ \mathbf{A}\,\mathbf{x} \le \mathbf{b} \right\} \rightleftarrows \left\{ \mathbf{x} \in \mathbb{R}^d_{\ge 0} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\}$

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For $t \in \mathbb{Z}_{>0}$, let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$

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Quasi-polynomial – $c_d(t)\, t^d + c_{d-1}(t)\, t^{d-1} + \cdots + c_0(t)$ where $c_k(t)$ are periodic

Theorem (Ehrhart 1967) If $\mathcal{P}$ is a rational polytope, then...

- $L_{\mathcal{P}}(t)$ and $L_{\mathcal{P}^\circ}(t)$ are quasi-polynomials in t of degree $\dim \mathcal{P}$
- If $\mathcal{P}$ has integer vertices, then $L_{\mathcal{P}}$ and $L_{\mathcal{P}^\circ}$ are polynomials
- Leading term: $\operatorname{vol}(P)$ (suitably normalized)
- $L_P(0) = \chi(P)$
- (Macdonald 1970) $L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)$

(Weak) semimagic squares revisited

$$H_n(t) = L_{\mathcal{B}_n}(t) = \# \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \in \mathbb{Z}_{\geq 0}^{n^2} : \begin{array}{l} \sum_j x_{jk} = t \\ \sum_k x_{jk} = t \end{array} \right\}$$

$$L_{\mathcal{B}_n^\circ}(t) = \# \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \in \mathbb{Z}_{> 0}^{n^2} : \begin{array}{l} \sum_j x_{jk} = t \\ \sum_k x_{jk} = t \end{array} \right\}$$

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$L_{\mathcal{B}_n^\circ}(t) = L_{\mathcal{B}_n}(t-n)$, so by Ehrhart-Macdonald reciprocity (Ehrhart, Stanley 1973)

$$H_n(-n-t) = (-1)^{(n-1)^2} \, H_n(t)$$

$$H_n(-1) = \cdots = H_n(-n+1) = 0 \; .$$

Computation of Ehrhart (quasi-)polynomials

- Pommersheim (1993): 3-dimensional tetrahedra – connection to Dedekind sum

$$\sum_{k=1}^{b-1} \cot \frac{\pi k a}{b} \cot \frac{\pi k}{b}$$

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- Formulas by Danilov, Brion-Vergne, Kantor-Khovanskii-Puklikov, Diaz-Robins, Chen, Baldoni-DeLoera-Szenes-Vergne, Lasserre-Zeron, . . .

Euler's generating function

$$\mathcal{P} := \left\{ \mathbf{x} \in \mathbb{R}^d_{\geq 0} : \ \mathbf{A}\,\mathbf{x} = \mathbf{b} \right\} \qquad \mathbf{A} = \begin{pmatrix} | & | & & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_d \\ | & | & & | \end{pmatrix}$$

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$L_{\mathcal{P}}(t)$ equals the coefficient of $\mathbf{z}^{t\mathbf{b}} := z_1^{tb_1} \cdots z_m^{tb_m}$ of the function

$$\frac{1}{\left(1 - \mathbf{z}^{\mathbf{c}_1}\right) \cdots \left(1 - \mathbf{z}^{\mathbf{c}_d}\right)}$$

expanded as a power series centered at $\mathbf{z} = 0$.

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Proof Expand each factor into a geometric series. ⬠

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Equivalently,

$$L_{\mathcal{P}}(t) = \operatorname{const} \frac{1}{(1 - \mathbf{z}^{\mathbf{c}_1}) \cdots (1 - \mathbf{z}^{\mathbf{c}_d})\, \mathbf{z}^{t\mathbf{b}}}$$

Partition functions and the Frobenius problem

Restricted partition function for $A = \{a_1, \dots, a_d\}$

$$p_A(t) = \# \left\{ (m_1, \dots, m_d) \in \mathbb{Z}_{\geq 0}^d : \ m_1 a_1 + \cdots + m_d a_d = t \right\}$$

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Hence $p_A(t)$ is a quasipolynomial in t of degree $d-1$ and period $\operatorname{lcm}(a_1, \ldots, a_d)$.

$$p_A(t) = \operatorname{const} \frac{1}{(1 - z^{a_1})(1 - z^{a_2}) \cdots (1 - z^{a_d}) z^t}$$

Fourier-Dedekind sum

defined for $c_1, \ldots, c_d \in \mathbb{Z}$ relatively prime to $c \in \mathbb{Z}$ and $n \in \mathbb{Z}$

$$\sigma_n\left(c_1, \ldots, c_d; c\right) = \frac{1}{c} \sum_{\lambda^c = 1 \neq \lambda} \frac{\lambda^n}{\left(1 - \lambda^{c_1}\right) \cdots \left(1 - \lambda^{c_d}\right)}$$

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Theorem If $a_1, \ldots, a_d$ are pairwise relatively prime then

$$p_A(t) = P_A(t) + \sum_{j=1}^{d} \sigma_{-t}(a_1, \ldots, \hat{a}_j, \ldots, a_d; a_j)$$

where

$$P_A(t) = \frac{1}{a_1 \cdots a_d} \sum_{m=0}^{d-1} \frac{(-1)^m}{(d-1-m)!} \sum_{k_1+\cdots+k_d=m} a_1^{k_1} \cdots a_d^{k_d} \frac{B_{k_1} \cdots B_{k_d}}{k_1! \cdots k_d!} \, t^{d-1-m}$$

Examples of Fourier-Dedekind sums

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Corollaries

- Pommersheim formulas
- Ehrhart quasipolynomials of all rational polygons $(d = 2)$ can be computed using Dedekind-Rademacher sums

Corollaries due to Ehrhart theory

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- If $0 < t < a_1 + \cdots + a_d$ then

$$\sum_{j=1}^{d} \sigma_t(a_1, \ldots, \hat{a}_j, \ldots, a_d; a_j) = -P_A(t)$$

(Specializes to reciprocity laws for generalized Dedekind sums due to Rademacher and Gessel)

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- $\displaystyle\sum_{j=1}^{d} \sigma_0(a_1, \ldots, \hat{a}_j, \ldots, a_d; a_j) = 1 - P_A(0)$

(Equivalent to Zagier's higher dimensional Dedekind sums reciprocity law)

Back to Birkhoff...

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$$= \left\{ \mathbf{x} \in \mathbb{R}_{\geq 0}^{n^2} : \ \mathbf{A}\,\mathbf{x} = \mathbf{1} \right\}$$

with

$$\mathbf{A} = \begin{pmatrix} 1 & \cdots & 1 & & & & & & & \\ & & & 1 & \cdots & 1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & 1 & \cdots & 1 \\ 1 & & & 1 & & & & 1 & & \\ & \ddots & & & \ddots & & \cdots & & \ddots & \\ & & 1 & & & 1 & & & & 1 \end{pmatrix}$$

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$$\begin{aligned} H_n(t) &= \operatorname{const} \frac{(z_1 \cdots z_{2n})^{-t}}{(1 - z_1 z_{n+1})(1 - z_1 z_{n+2}) \cdots (1 - z_n z_{2n})} \\ &= \operatorname{const}_{\mathbf{z}} \left((z_1 \cdots z_n)^{-t} \left(\operatorname{const}_w \frac{w^{-t-1}}{(1 - z_1 w) \cdots (1 - z_n w)} \right)^n \right) \\ &= \operatorname{const} \left((z_1 \cdots z_n)^{-t} \left(\sum_{k=1}^n \frac{z_k^{t+n-1}}{\prod_{j \neq k} (z_k - z_j)} \right)^n \right) \end{aligned}$$

$n = 3$

$$H_3(t) = \text{const}(z_1 z_2 z_3)^{-t} \times$$
$$\left(\frac{z_1^{t+2}}{(z_1 - z_2)(z_1 - z_3)} + \frac{z_2^{t+2}}{(z_2 - z_1)(z_2 - z_3)} + \frac{z_3^{t+2}}{(z_3 - z_1)(z_3 - z_2)} \right)^3$$

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$$= \operatorname{const} \frac{z_1^{2t+6} z_2^{-t} z_3^{-t}}{(z_1 - z_2)^3 (z_1 - z_3)^3} - 3 \operatorname{const} \frac{z_1^{t+4} z_2^2 z_3^{-t}}{(z_1 - z_2)^3 (z_1 - z_3)^2 (z_2 - z_3)}$$

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$$= \text{const}_{z_1} \left(z_1^{2t+6} \left(\text{const}_z \frac{z^{-t}}{(z_1 - z)^3} \right)^2 \right) - 3 \, \text{const} \frac{z_1^{t+4} z_3^{-t}}{(z_1 - z_3)^5}$$

$$= \binom{t+2}{2}^2 - 3 \binom{t+3}{4} = \frac{1}{8} t^4 + \frac{3}{4} t^3 + \frac{15}{8} t^2 + \frac{9}{4} t + 1$$

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$$H_3(t) = \operatorname{const}(z_1 z_2 z_3)^{-t} \times$$
$$\left(\frac{z_1^{t+2}}{(z_1 - z_2)(z_1 - z_3)} + \frac{z_2^{t+2}}{(z_2 - z_1)(z_2 - z_3)} + \frac{z_3^{t+2}}{(z_3 - z_1)(z_3 - z_2)} \right)^3$$
$$= \operatorname{const} \frac{z_1^{2t+6} z_2^{-t} z_3^{-t}}{(z_1 - z_2)^3 (z_1 - z_3)^3} - 3 \operatorname{const} \frac{z_1^{t+4} z_2^2 z_3^{-t}}{(z_1 - z_2)^3 (z_1 - z_3)^2 (z_2 - z_3)}$$
$$= \operatorname{const}_{z_1} \left(z_1^{2t+6} \left(\operatorname{const}_z \frac{z^{-t}}{(z_1 - z)^3} \right)^2 \right) - 3 \operatorname{const} \frac{z_1^{t+4} z_3^{-t}}{(z_1 - z_3)^5}$$
$$= \binom{t+2}{2}^2 - 3 \binom{t+3}{4} = \frac{1}{8} t^4 + \frac{3}{4} t^3 + \frac{15}{8} t^2 + \frac{9}{4} t + 1$$

$$\Longrightarrow \qquad \operatorname{vol} \mathcal{B}_3 = 3^2 \cdot \frac{1}{8} = \frac{9}{8}$$

$n = 4$

After computing five constant terms . . .

$$H_4(t) = \binom{t+3}{3}^3 + 6\left(2t^2+5t+1\right)\binom{t+5}{7} - 24\,(t+4)\binom{t+5}{8}$$
$$+12(t+1)\binom{t+6}{8} - 4\binom{2t+8}{9} - 48\binom{t+5}{9} + 12\binom{t+7}{9}$$
$$= \frac{11}{11340}t^9 + \frac{11}{630}t^8 + \frac{19}{135}t^7 + \frac{2}{3}t^6 + \frac{1109}{540}t^5$$
$$+\frac{43}{10}t^4 + \frac{35117}{5670}t^3 + \frac{379}{63}t^2 + \frac{65}{18}t + 1$$

$$\operatorname{vol}\mathcal{B}_4 = 4^3 \cdot \frac{11}{11340} = \frac{176}{2835}$$

The relative volume of the fundamental domain of the sublattice of $\mathbb{Z}^{n^2}$ in the affine space spanned by $\mathcal{B}_n$ is n^{n-1}.

General n

$$H_n(t) = \text{const}(z_1 \cdots z_n)^{-t} \times$$
$$\sum_{m_1+\cdots+m_n=n}^{*} \binom{n}{m_1, \ldots, m_n} \prod_{k=1}^{n} \left(\frac{z_k^{t+n-1}}{\prod_{j \neq k}(z_k - z_j)} \right)^{m_k}$$

where $\sum^*$ denotes that we only sum over those n-tuples of non-negative integers satisfying

$$m_1 + \cdots + m_n = n$$

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$$m_1 + \cdots + m_r > r \quad \text{if} \quad 1 \leq r < n \ .$$

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Computational concerns:

- # terms in the sum is $C_{n-1} = \dfrac{(2n-2)!}{(n)!(n-1)!}$
- Iterated constant-term computation

Analytic speed-up tricks

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- Factor constant-term computation if some of the variables appear in a symmetric fashion
- If only interested in $\operatorname{vol} \mathcal{B}_n$, we may dispense a particular constant term if it does not contribute to leading term of H_n.

Volumes of B_n

n	$\operatorname{vol} \mathcal{B}_n$
1	1
2	2
3	$9/8$
4	$176/2835$
5	$23590375/167382319104$
6	$9700106723/1319281996032 \cdot 10^6$
7	$\dfrac{77436678274508929033}{137302963682235238399868928 \cdot 10^8}$
8	$\dfrac{5562533838576105333259507434329}{12589036260095477950081480942693339803308928 \cdot 10^{10}}$
9	$\dfrac{559498129702796022246895686372766052475496691}{92692623409952636498965146712806984296051951329202419606108477153345536 \cdot 10^{14}}$

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$$\frac{727291284016786420977508457990121862548823260052557333386607889}{82816086010676685512567631879687272934462246353308942267798072138805573995627029375088350489282084864 \cdot 10^7}$$

Computation times

n	computing time for $\operatorname{vol} \mathcal{B}_n$
1	$<$ .01 sec
2	$<$ .01 sec
3	$<$ .01 sec
4	$<$ .01 sec
5	$<$ .01 sec
6	.18 sec
7	15 sec
8	54 min
9	317 hr
10	6160 d

(scaled to a 1GHz processor running Linux)

Coefficients and roots of Ehrhart polynomials

Lattice (convex) polytope $\mathcal{P}$ – convex hull of finitely many points in $\mathbb{Z}^d$

Then $L_{\mathcal{P}}(t) = c_d\, t^d + \cdots + c_0$ is a polynomial in t.

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Theorem (Stanley 1980) The generating function $\sum_{t \ge 0} L_{\mathcal{P}}(t)\, x^t$ can be written in the form $\frac{f(x)}{(1-x)^{d+1}}$, where $f(x)$ is a polynomial of degree at most d with nonnegative integer coefficients.

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- The inequalities $f(x) \ge 0$ and $c_{d-1} > 0$ are currently the sharpest constraints on Ehrhart coefficients. Are there others?

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(1) The roots of Ehrhart polynomials of lattice d-polytopes are bounded in norm by $1 + (d+1)!$.

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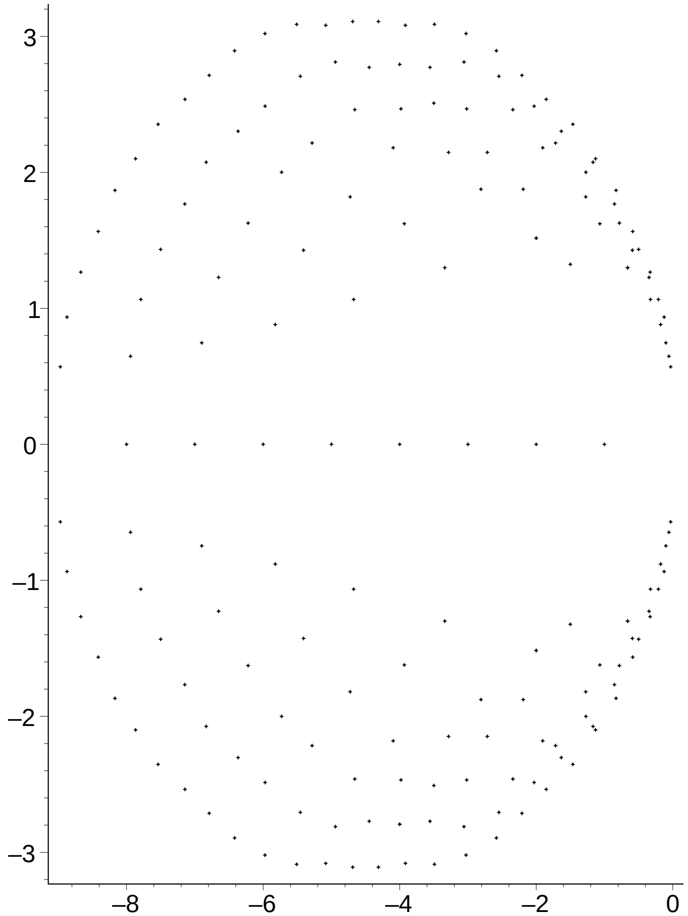
Conjecture: All roots α satisfy $-d \leq \text{Re } \alpha \leq d-1$.

Roots of the Birkhoff polytopes



Roots of some tetrahedra