Discre~~e~~te Volume Computations for Polytopes: An Invitation to Ehrhart Theory

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Meet my friends . . .

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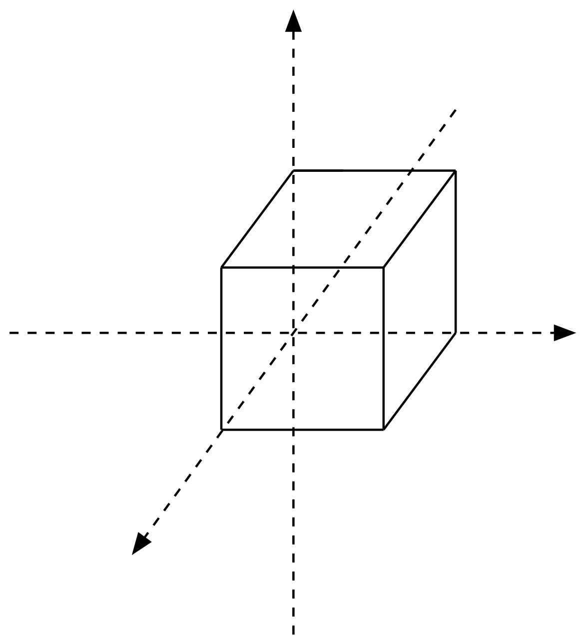
Example: the 3-dimensional unit cube . . .

$$\left\{ \begin{array}{c} (x, y, z) \in \mathbb{R}^3 : \\ 0 \le x \le 1 \\ 0 \le y \le 1 \\ 0 \le z \le 1 \end{array} \right\}$$



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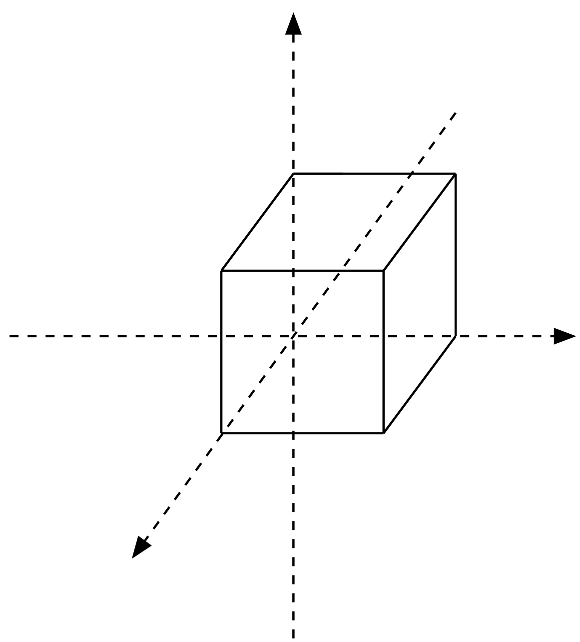

$$\begin{array}{c} (0,0,0) \\ (1,0,0) \\ (0,1,0) \\ (0,0,1) \\ (1,1,0) \\ (1,0,1) \\ (0,1,1) \\ (1,1,1) \end{array}$$

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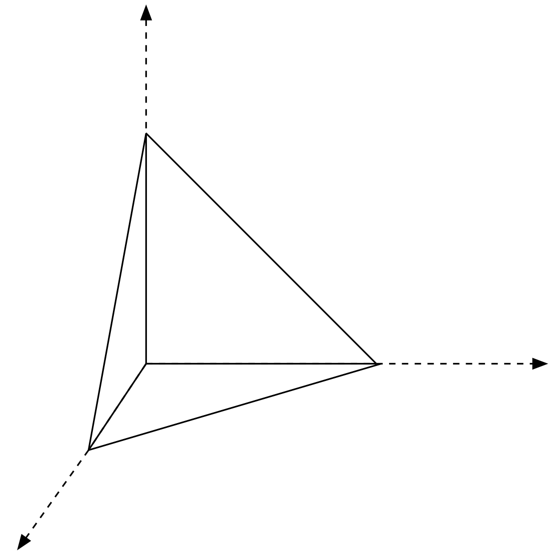

The standard simplex

$$\Delta = \left\{ \mathbf{x} \in \mathbb{R}^d : \, x_1 + x_2 + \cdots + x_d \le 1, \, x_j \ge 0 \right\}$$
$$= \operatorname{conv} \left\{ (0,0,\dots,0), (1,0,0,\dots,0), (0,1,0,\dots,0), \dots, (0,0,\dots,0,1) \right\}$$

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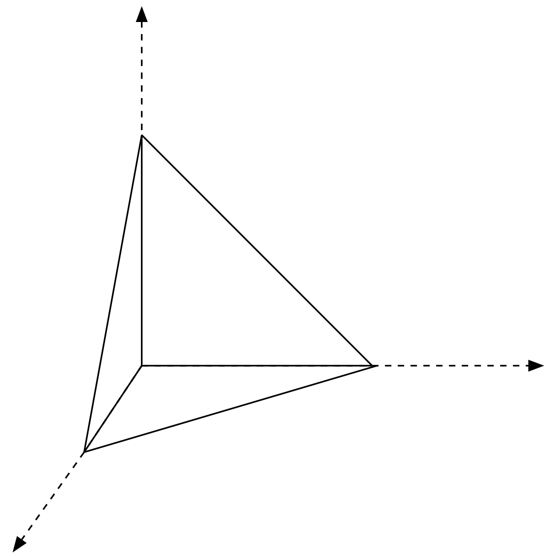

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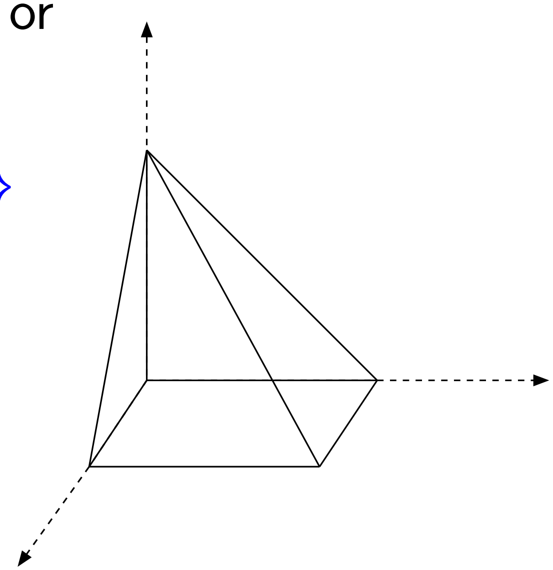
The pyramid over the $(d-1)$-dimensional unit cube $\square$: the convex hull of $\square$ (lifted into dimension d) and $(0,0,\dots,0,1)$ or



$$\mathrm{Pyr} = \left\{ \begin{array}{c} (x_1, x_2, \dots, x_d) \in \mathbb{R}^d : \\ 0 \le x_1, x_2, \dots, x_{d-1} \le 1 - x_d \le 1 \end{array} \right\}$$

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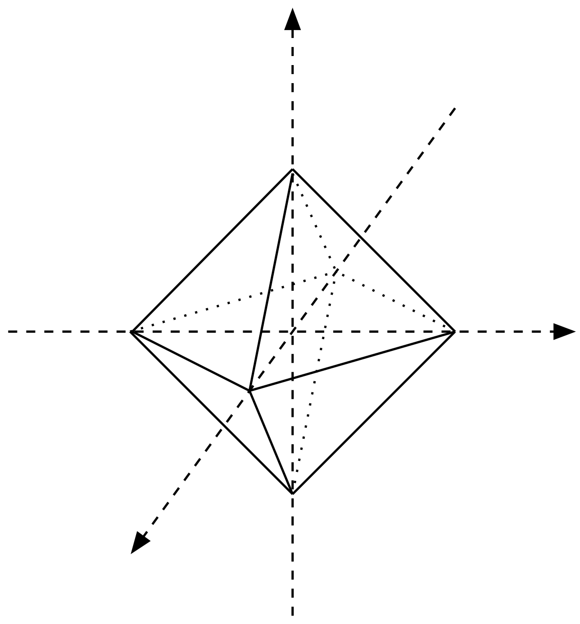
The cross-polytope

$$\diamond = \left\{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : \ |x_1| + |x_2| + \cdots + |x_d| \le 1 \right\}$$
$$= \operatorname{conv} \left\{ (\pm 1, 0, \ldots, 0), (0, \pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1) \right\}$$



A Plug For Great, Free Software

YOU should check out Ewgenij Gawrilow and Michael Joswig's polymake

www.math.tu-berlin.de/polymake

Today's Goal

Given a lattice polytope $\mathcal{P}$ (i.e., the extreme points are in $\mathbb{Z}^d$), compute its (continuous) volume

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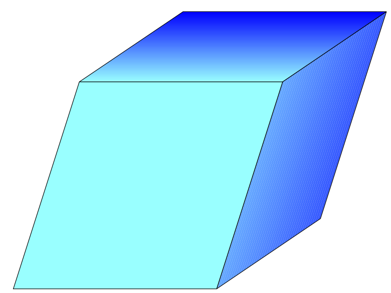

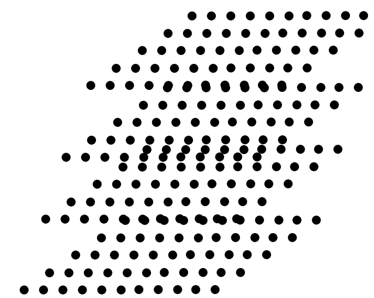

Approach: Discretize the problem . . .

$$\operatorname{vol} \mathcal{P} = \lim_{t \to \infty} \frac{\# \left(\mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d \right)}{t^d} \ .$$

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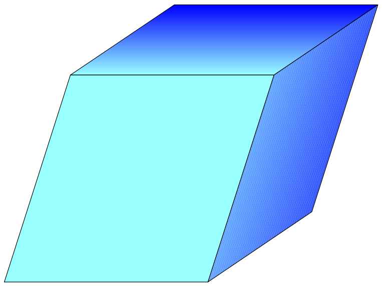

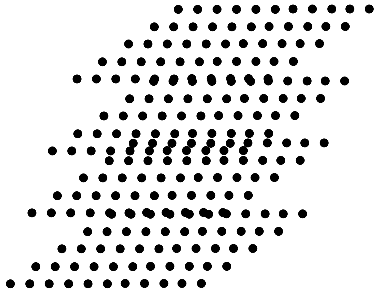

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For a positive integer t we define the discrete volume of $\mathcal{P}$ as

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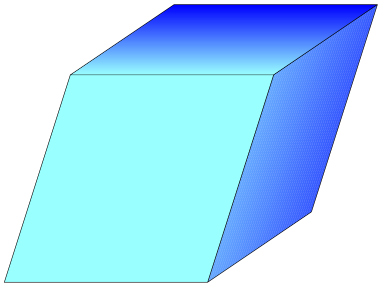

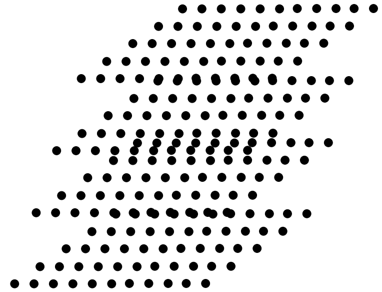

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Today's real goal: Given a lattice polytope $\mathcal{P}$, compute $L_{\mathcal{P}}(t)$.

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- Many discrete problems in various mathematical areas are linear problems, thus they ask for the discrete volume of a polytope in disguise.
- Polytopes are cool.

A Warm-Up Example

Let's consider the unit square $\square = \{(x,y) \in \mathbb{R}^2 : \, 0 \leq x, y \leq 1\}$

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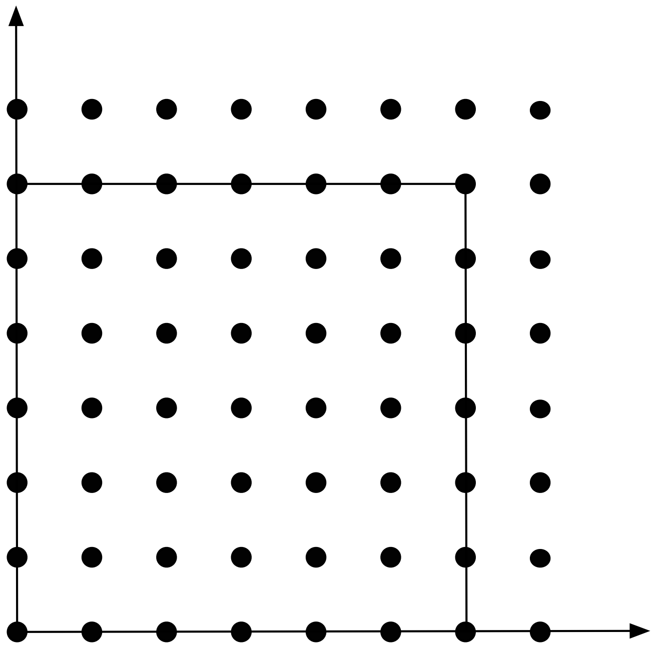

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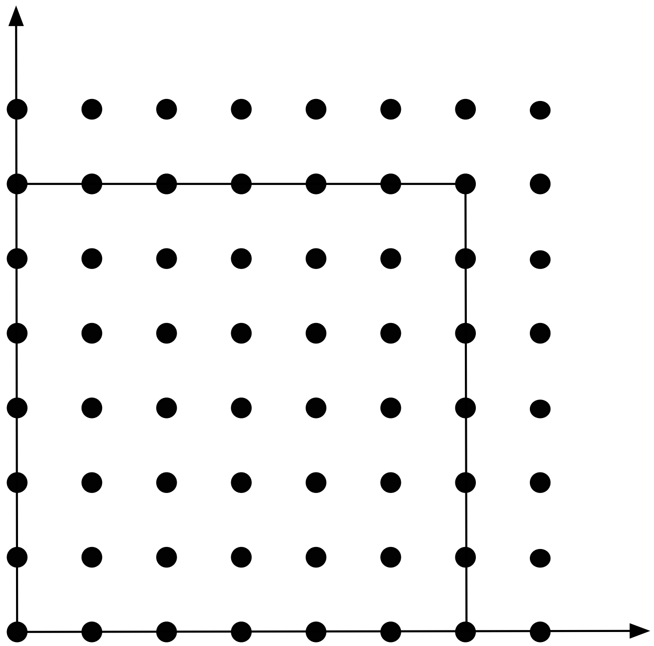

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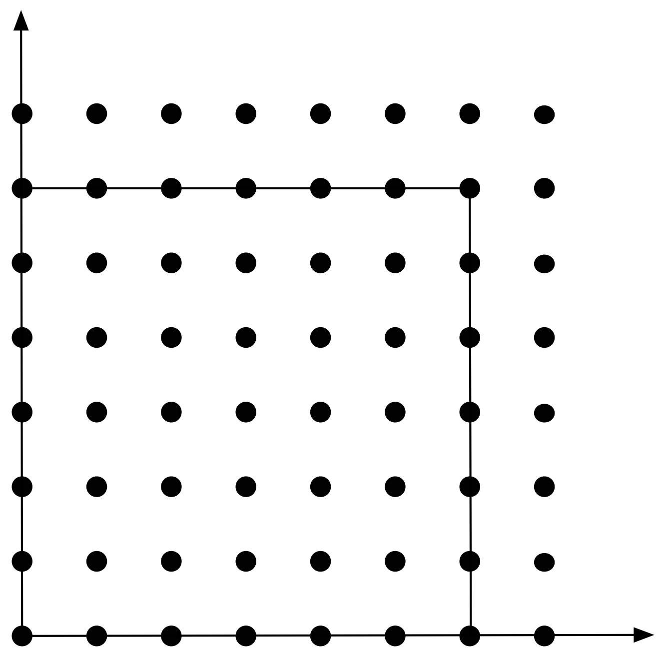

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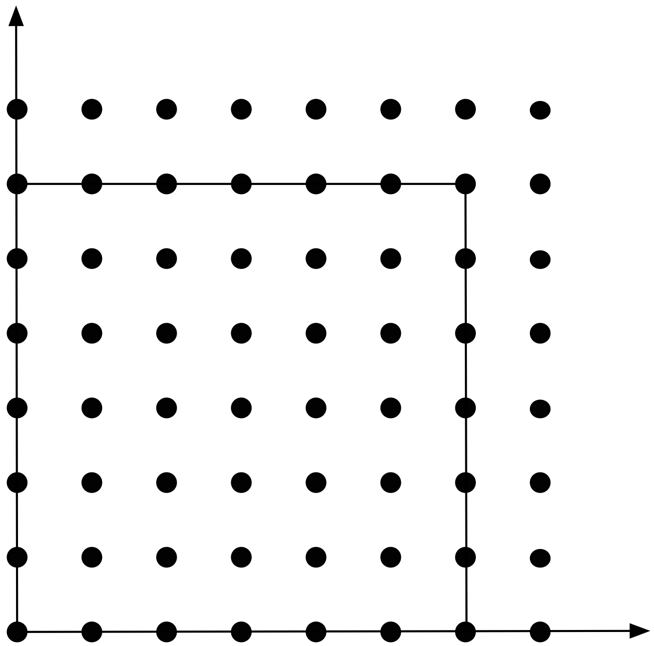

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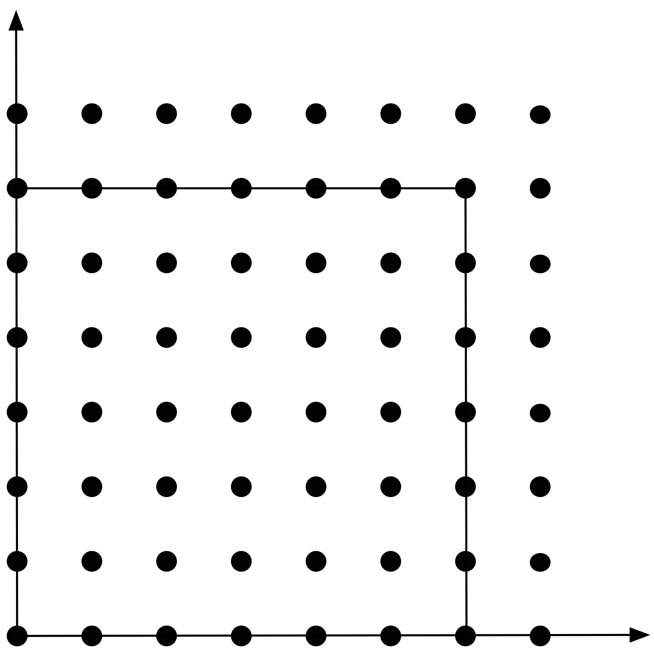

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$$L_{\square^\circ}(t) = \# \left(\square^\circ \cap \frac{1}{t}\mathbb{Z}^2 \right) = (t-1)^2 = t^2 - 2t + 1 = L_\square(-t)$$

A Warm-Up Example in General Dimension

For the unit d-cube $\square = \left\{ (x_1, x_2, \dots, x_d) \in \mathbb{R}^d : \, 0 \le x_j \le 1 \right\}$ we obtain the analogous formulas

$$L_{\square}(t) = (t+1)^d \qquad \text{and} \qquad L_{\square^\circ}(t) = (t-1)^d .$$

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$$L_\square(t) = \sum_{k=0}^{d} \binom{d}{k} t^k , \qquad \qquad \operatorname{vol}(\square) = 1$$

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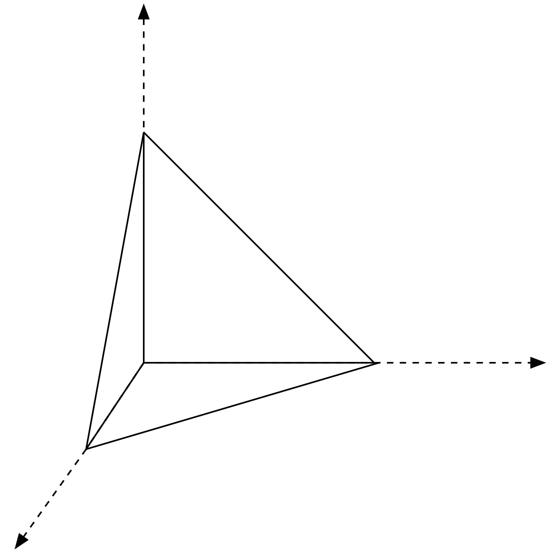
$$L_{\Box}(-t) = (-1)^d L_{\Box^\circ}(t) \ .$$

The Standard Simplex



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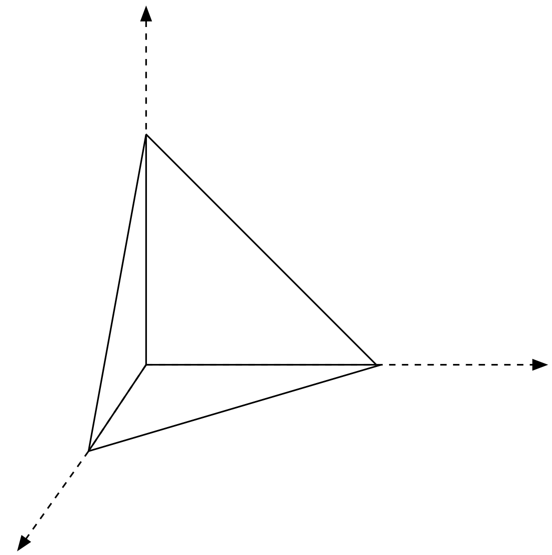

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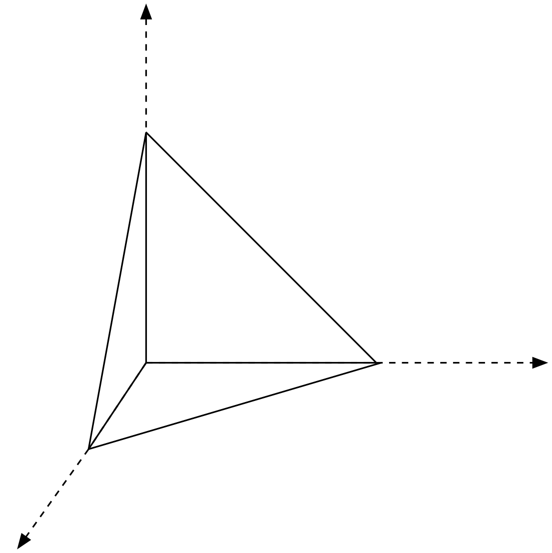

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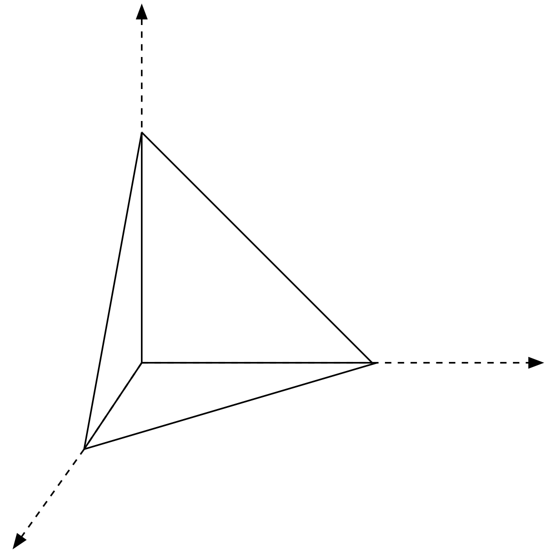

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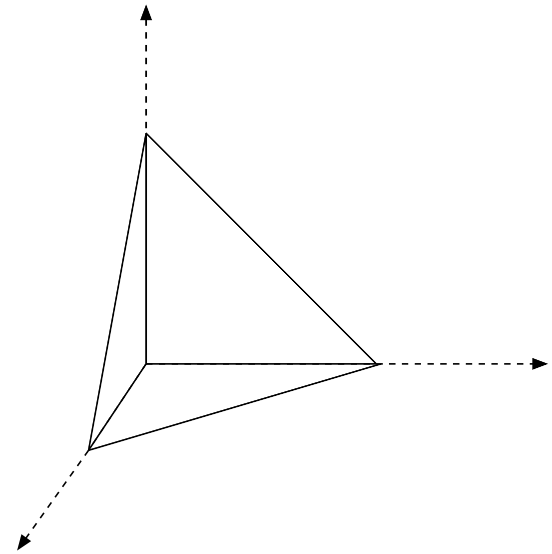

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$$= \binom{t+d}{d} = \frac{(t+d)(t+d-1)\cdots(t+1)}{d!},$$

a polynomial in t with leading coefficient $\operatorname{vol}(\Delta) = \frac{1}{d!}$.

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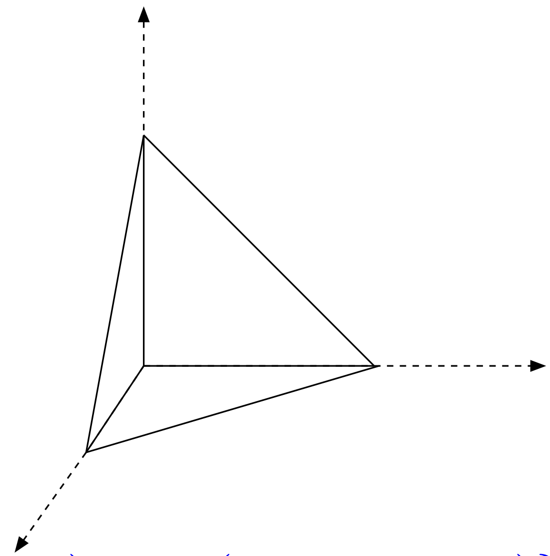

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Incidentally, $$L_\Delta(t) = \frac{1}{d!} \sum_{k=0}^{d} (-1)^{d-k} \operatorname{stirl}(d+1, k+1) \, t^k \, ,$$

where $\operatorname{stirl}(n, j)$ are the Stirling numbers of the first kind.

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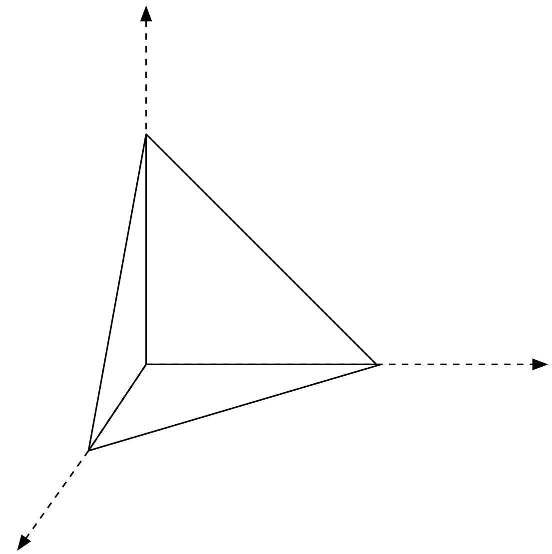

The interior of the d-simplex,

$$\Delta^\circ = \left\{ \mathbf{x} \in \mathbb{R}^d : \, x_1 + x_2 + \cdots + x_d < 1, \, x_j > 0 \right\}$$

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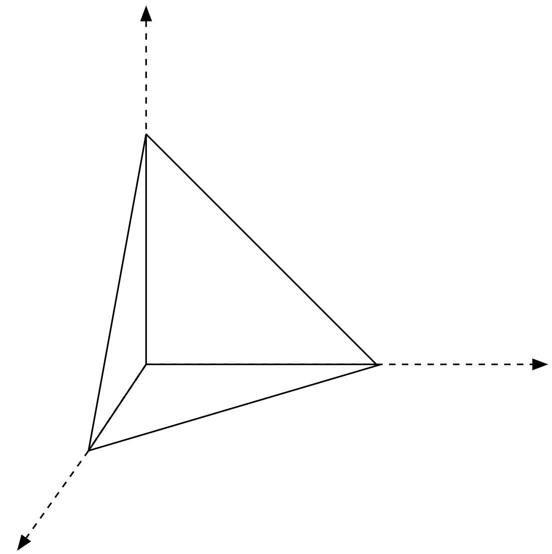

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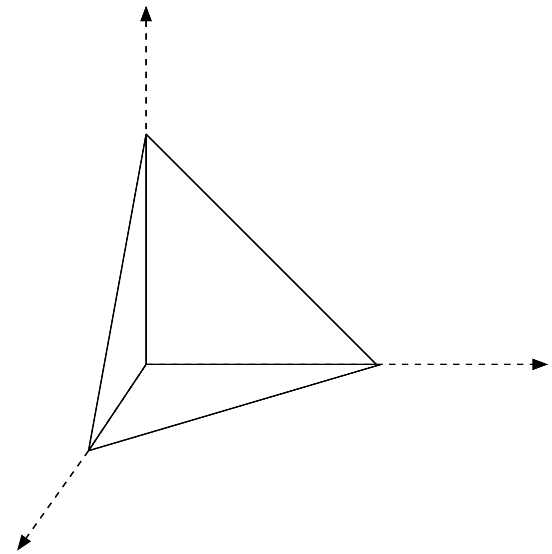

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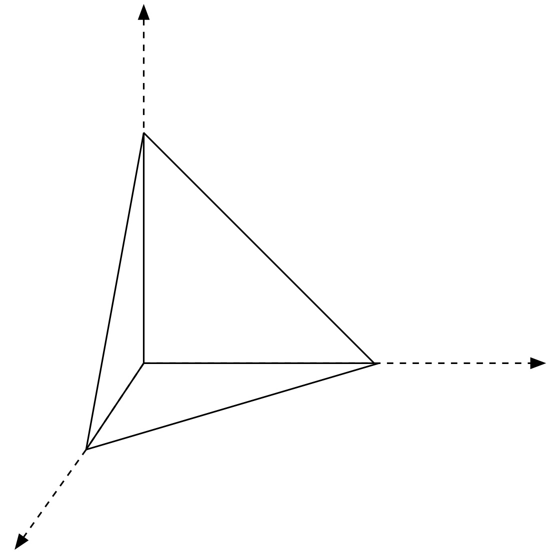

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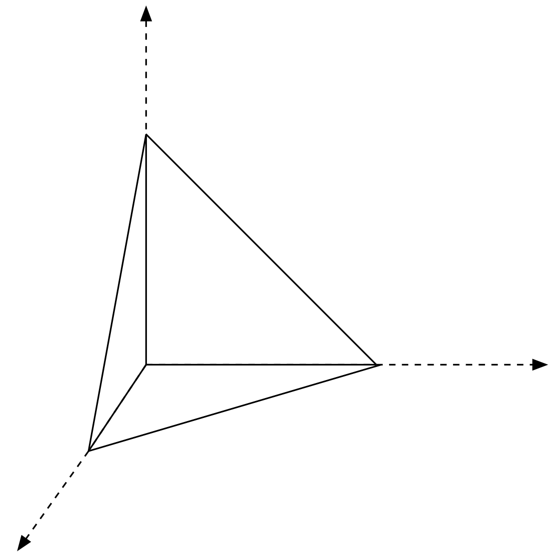

The interior of the d-simplex,

$$\Delta^\circ = \left\{ \mathbf{x} \in \mathbb{R}^d : \, x_1 + x_2 + \cdots + x_d < 1, \, x_j > 0 \right\},$$

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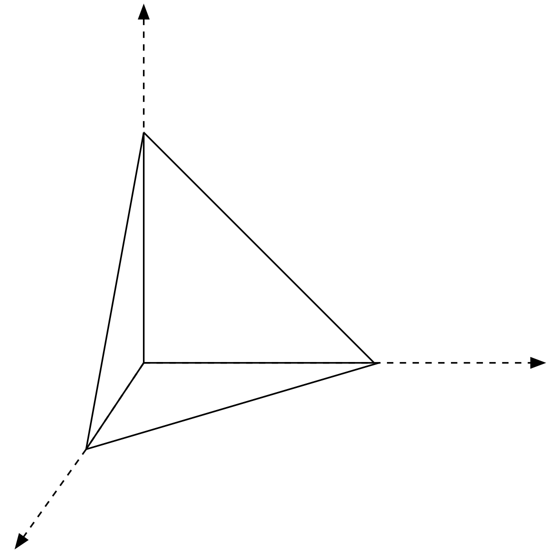

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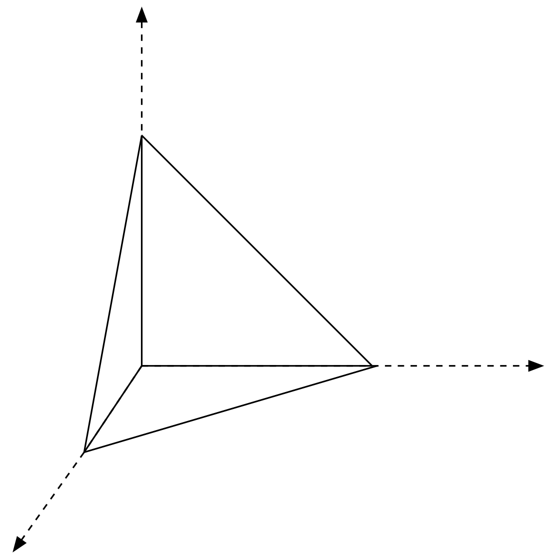

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Generating Functions

The discrete volume $L_\Delta(t) = \binom{t+d}{d}$ of the standard d-simplex comes with the friendly generating function

$$\sum_{t \geq 0} \binom{t+d}{d} z^t = \frac{1}{(1-z)^{d+1}} \,.$$

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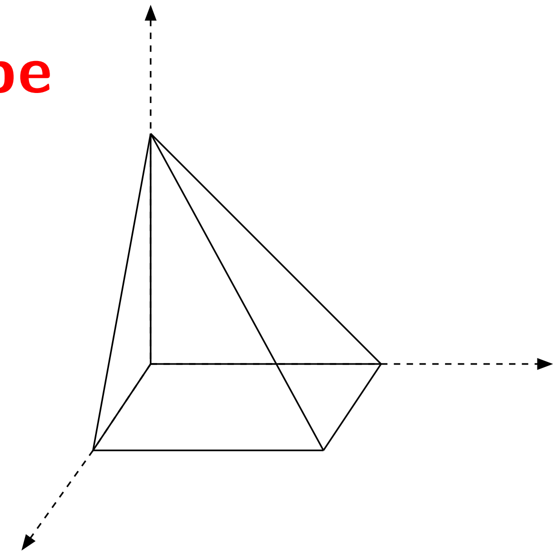
Pyramids over the Unit Cube



Recall the pyramid over the $(d-1)$-dimensional unit cube $\Box$: the convex hull of $\Box$ (lifted into dimension d) and $(0,0,\ldots,0,1)$ or

$$\text{Pyr} = \left\{ \begin{array}{c} (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : \\ 0 \le x_1, x_2, \ldots, x_{d-1} \le 1 - x_d \le 1 \end{array} \right\}.$$

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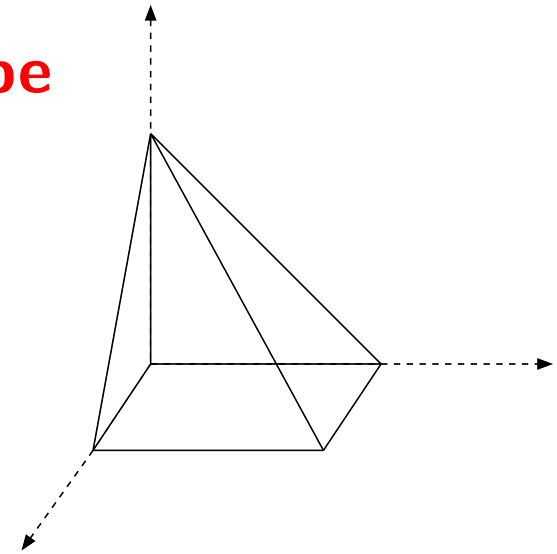

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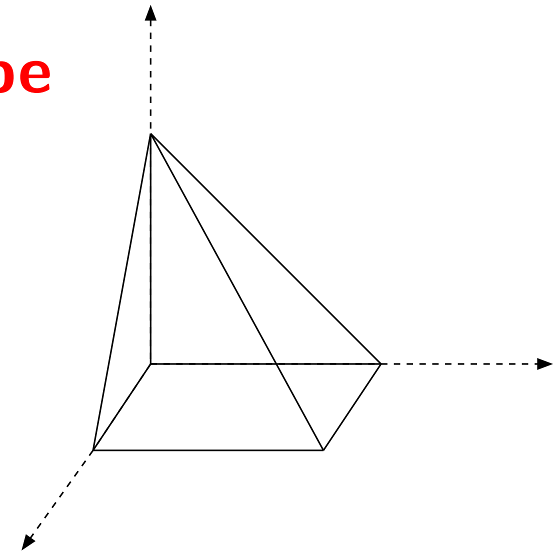

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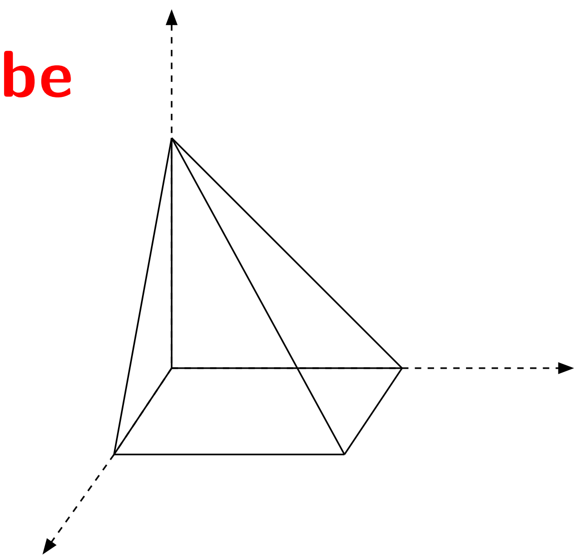

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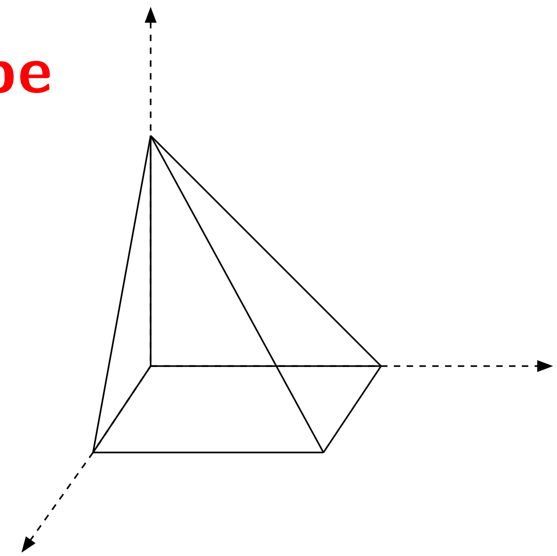

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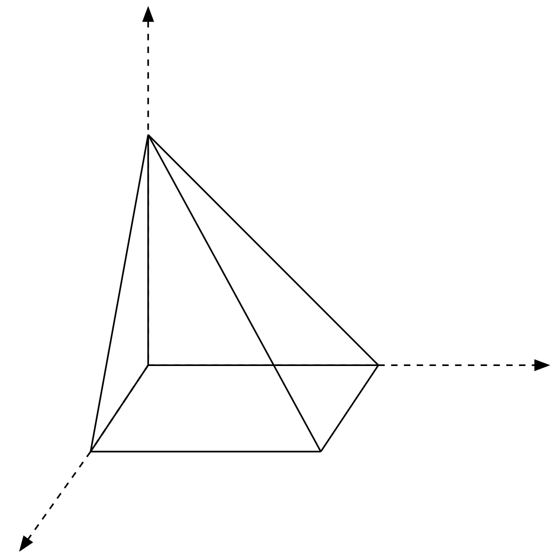
where $B_d(x)$ denotes the d'th Bernoulli polynomial. The Bernoulli polynomials are monic, and so $\mathrm{vol}(\mathrm{Pyr}) = \frac{1}{d}$.

Pyramids over the Unit Cube

The Bernoulli polynomials are defined through

$$\frac{z\,e^{xz}}{e^z - 1} = \sum_{k \geq 0} \frac{B_k(x)}{k!}\,z^k$$



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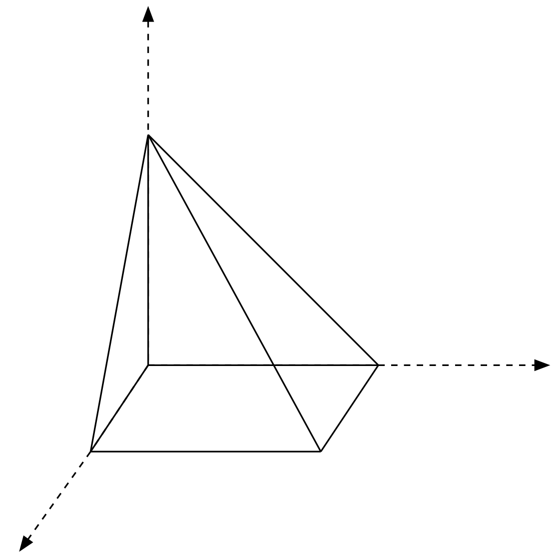
$$B_d(1-x) = (-1)^d B_d(x) \; .$$



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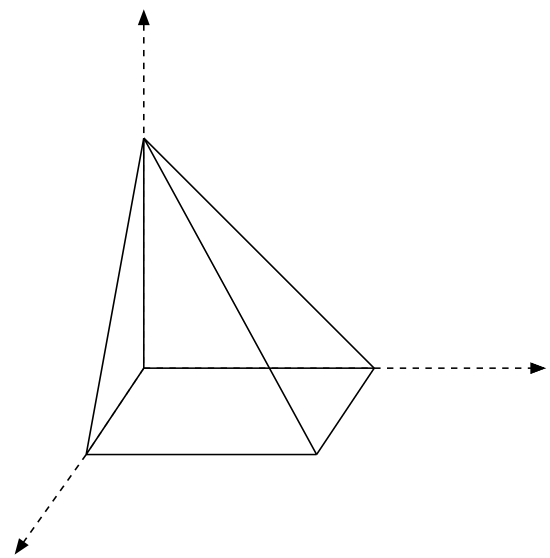

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The discrete volume of the interior of Pyr can be computed similarly:

$$L_{\mathrm{Pyr}^\circ}(t) = \frac{1}{d} \left(B_d(t-1) - B_d(0) \right),$$

which gives

$$L_{\mathrm{Pyr}}(-t) = (-1)^d L_{\mathrm{Pyr}^\circ}(t) \; .$$

A Pyramid Exercise

If $\mathcal{P}$ is a $(d-1)$-dimensional lattice polytope, let $\mathrm{Pyr}(\mathcal{P})$ be the convex hull of $\mathcal{P}$ (lifted into dimension d) and the point $(0,0,\dots,0,1)$. Then

$$\mathrm{Ehr}_{\mathrm{Pyr}(\mathcal{P})}(z) = \frac{\mathrm{Ehr}_{\mathcal{P}}(z)}{1-z} \, .$$

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For example, for the pyramid over the unit $(d-1)$-cube, we obtain

$$\mathrm{Ehr}_{\mathrm{Pyr}(\square)}(z) = \frac{\sum_{k=1}^{d-1} A(d-1,k)\, z^{k-1}}{(1-z)^{d+1}} \, ,$$

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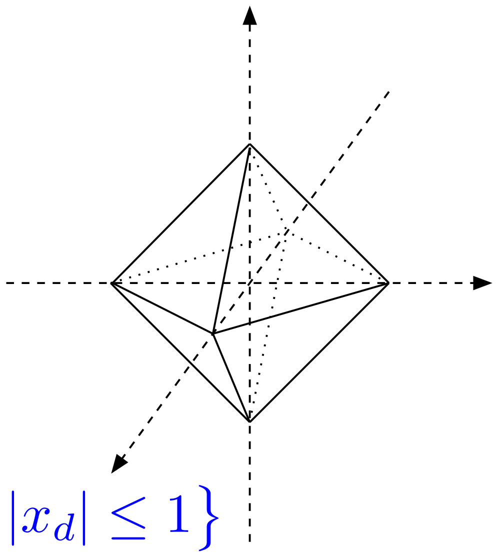
The Cross-Polytope



To compute the discrete volume of the cross-polytope

$$\begin{aligned} \Diamond &= \left\{ (x_1, x_2, \dots, x_d) \in \mathbb{R}^d : \ |x_1| + |x_2| + \cdots + |x_d| \le 1 \right\} \\ &= \operatorname{conv} \left\{ (\pm 1, 0, \dots, 0), (0, \pm 1, 0, \dots, 0), \dots, (0, \dots, 0, \pm 1) \right\}, \end{aligned}$$

we start with an exercise about bipyramids:

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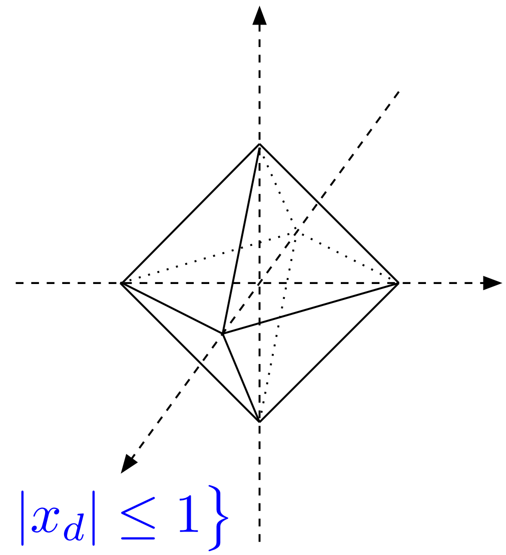

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If $\mathcal{P}$ is a $(d-1)$-dimensional lattice polytope, let $\operatorname{BiPyr}(\mathcal{P})$ be the convex hull of $\mathcal{P}$ (lifted into dimension d) and the points $(0, 0, \dots, 0, \pm1)$. Then

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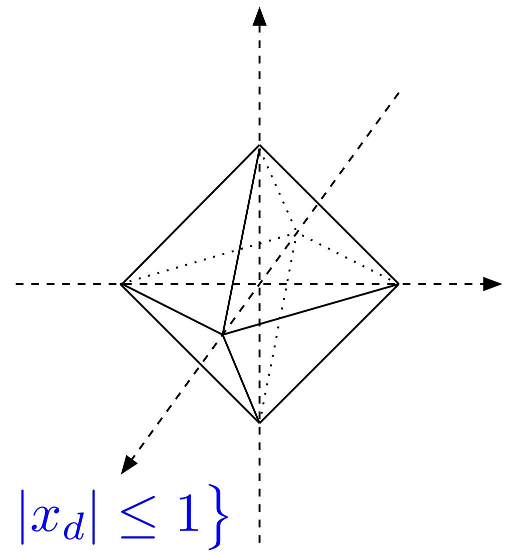

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For example, the d-dimensional cross-polytope $\diamond$ is the bipyramid over the $(d-1)$-dimensional cross-polytope.

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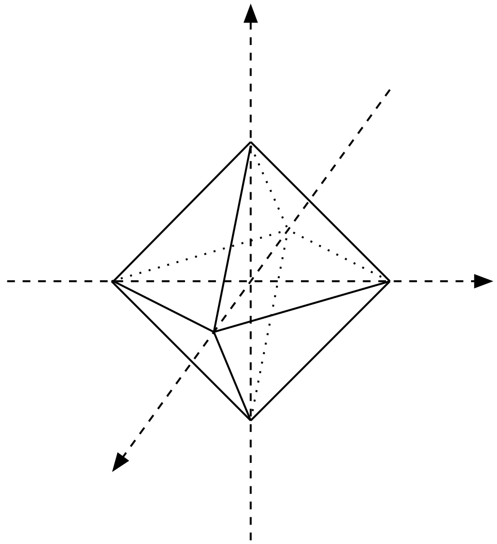

We thus recursively compute

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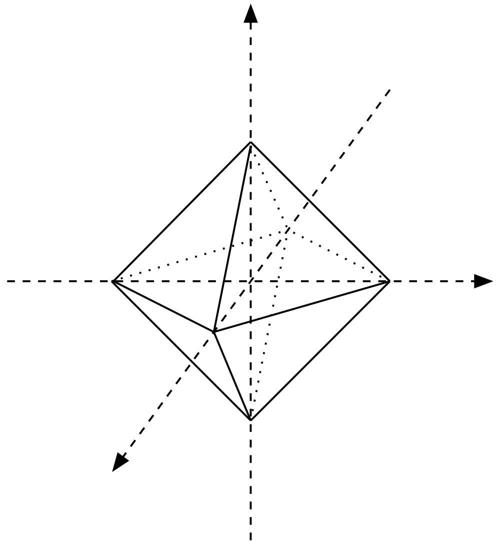

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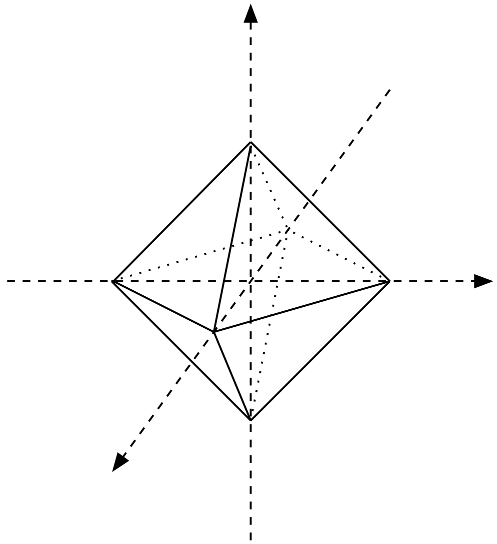

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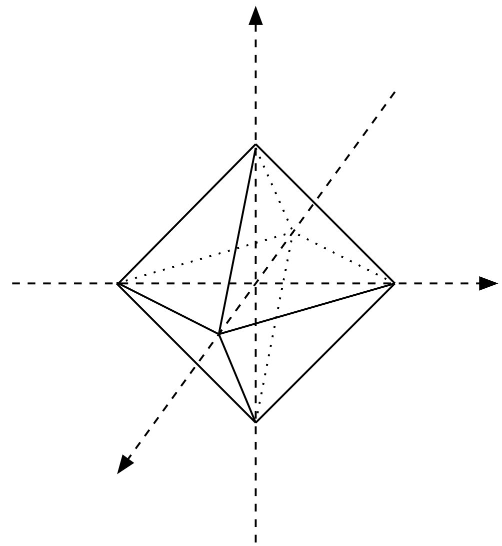

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Using the binomial reciprocity $\binom{m-1}{d} = (-1)^d \binom{-m+d}{d}$, we can see that

$$L_{\diamond}(-t) = (-1)^d L_{\diamond^\circ}(t) \ .$$

Pick's Theorem

For a lattice polygon $\mathcal{P}$ containing I interior and B boundary lattice point, Pick's Theorem tells us how to compute the area of P:

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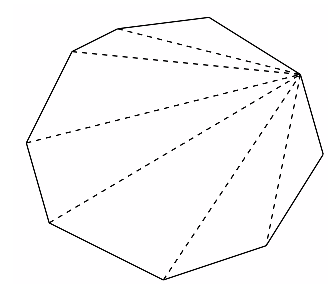
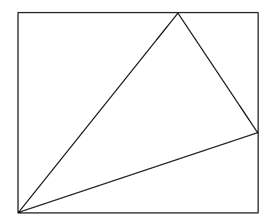
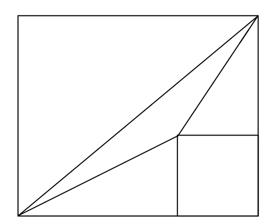


Do-it-yourself proof:

(1) Convince yourself that Pick's formula is "additive".





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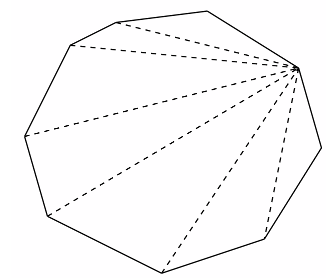

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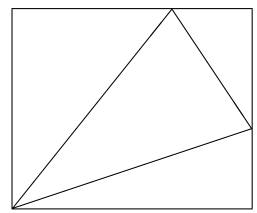
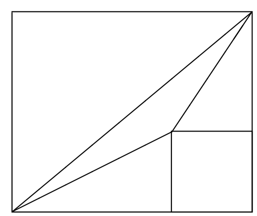


(2) Reduce to rectangles and right-angled triangles.



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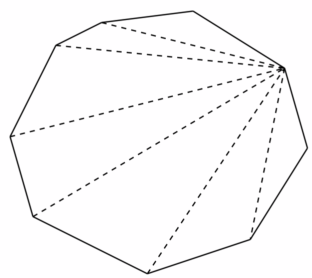

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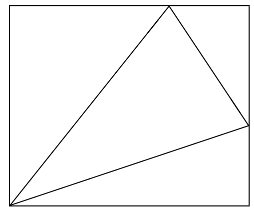
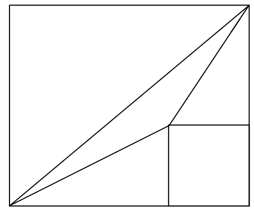


(2) Reduce to rectangles and right-angled triangles.

(3) Prove Pick's formula for these two cases.



Pick's Theorem Extended

$\mathcal{P}$ – lattice polygon with area A and B boundary lattice points

For a positive integer t, let $A(t)$ denote the area of $t\mathcal{P}$ and $B(t)$ the number of boundary lattice points of $t\mathcal{P}$.

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From this one easily obtains

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{\left(A - \frac{B}{2} + 1\right) z^2 + \left(A + \frac{B}{2} - 2\right) z + 1}{(1-z)^3} \; .$$

Ehrhart's Theorem

Theorem (Ehrhart 1962) Suppose $\mathcal{P}$ is a lattice polytope. Then $L_{\mathcal{P}}(t)$ and $L_{\mathcal{P}^\circ}(t)$ are polynomials in $t \in \mathbb{Z}_{>0}$ of degree $\dim \mathcal{P}$. Equivalently, $\mathrm{Ehr}_{\mathcal{P}}(z)$ and $\mathrm{Ehr}_{\mathcal{P}^\circ}(z)$ are rational functions of the form $\frac{h(z)}{(1-z)^{\dim \mathcal{P}+1}}$ for some polynomials $h(z)$.

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Theorem (Ehrhart–Macdonald 1971) The polynomials $L_{\mathcal{P}}(t)$ and $L_{\mathcal{P}^\circ}(t)$ satisfy the reciprocity relation

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t) \ .$$

If You Want To See More . . .

M. Beck & S. Robins

Computing the continuous discretely
Integer-point enumeration in polyhedra

To be published by Springer at the end of 2006

Electronic copy available at math.sfsu.edu/beck

Another Plug For Great, Free Software

YOU should check out Jesús De Loera et al's LattE

`www.math.ucdavis.edu/~latte`

and Sven Verdoolaege's barvinok

`freshmeat.net/projects/barvinok`

A Few Open Problems

- Choose $d+1$ of the 2^d vertices of the unit d-cube $\square$, and let $\mathcal{S}$ be the simplex defined by their convex hull.

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- Study the roots of Ehrhart polynomials of integral polytopes in a fixed dimension. Study the roots of the numerator of Ehrhart series.