Dedekind–Carlitz Polynomials as Lattice-Point Enumerators in Rational Polyhedra

Matthias Beck (San Francisco State University)

Christian Haase (Freie Universität Berlin)

Asia Matthews (Queen's University)

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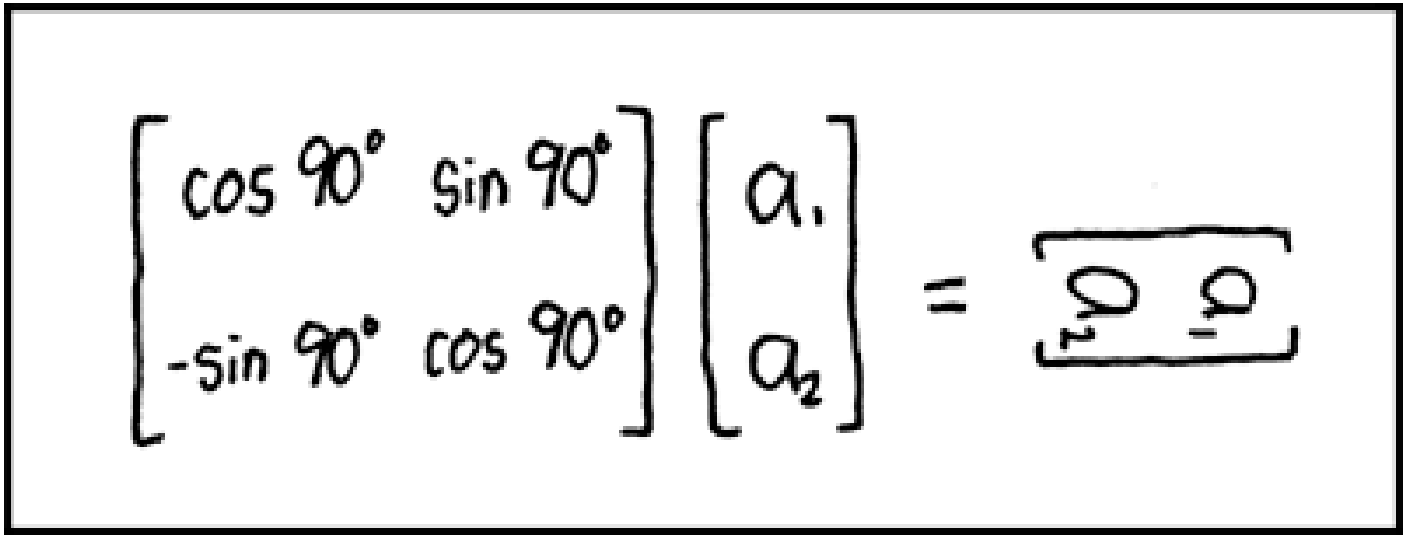
$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_2 & a_1 \end{bmatrix}$$



taken from `xkcd`, Randall Munroe's webcomic which "occasionally contains strong language (which may be unsuitable for children), unusual humor (which may be unsuitable for adults), and advanced mathematics (which may be unsuitable for liberal-arts majors)"

Dedekind Sums

Let

$$((x)) := \begin{cases} \{x\} - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}, \end{cases}$$

and define for positive integers a and b the Dedekind sum

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Since their introduction in the 1880's, the Dedekind sum and its generalizations have intrigued mathematicians from various areas such as analytic and algebraic number theory, topology, algebraic and combinatorial geometry, and algorithmic complexity.

Dedekind–Carlitz Polynomials

In the 1970's, Leonard Carlitz introduced the following polynomial generalization of the Dedekind sum:

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$$(v-1)\ \mathrm{c}\,(u, v; a, b) + (u-1)\ \mathrm{c}\,(v, u; b, a) = u^{a-1} v^{b-1} - 1\,.$$

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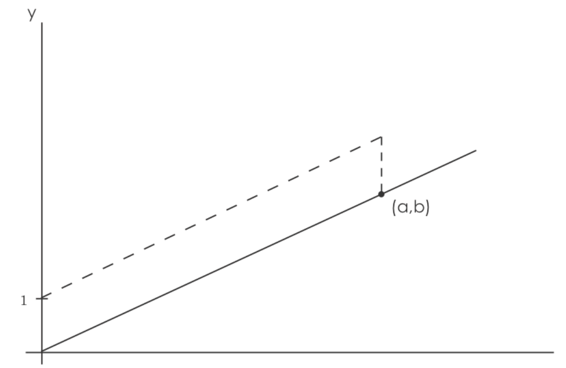
Applying $u\,\partial u$ twice and $v\,\partial v$ once gives Dedekind's reciprocity law

$$\mathrm{s}\,(a,b) + \mathrm{s}\,(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right).$$

Enter Polyhedral Geometry



Decompose the first quadrant $\mathbb{R}^2_{\geq 0}$ into the two cones

$$\mathcal{K}_1 = \{\lambda_1(0,1) + \lambda_2(a,b) : \lambda_1, \lambda_2 \geq 0\},$$
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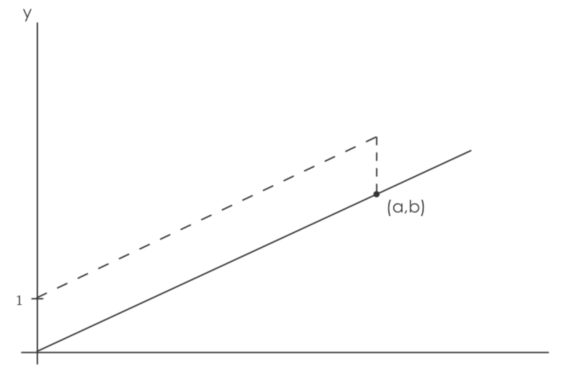

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Let's compute the integer-point transforms

$$\sigma_{\mathcal{K}_1}(u,v) := \sum_{(m,n)\in\mathcal{K}_1\cap\mathbb{Z}^2} u^m v^n$$

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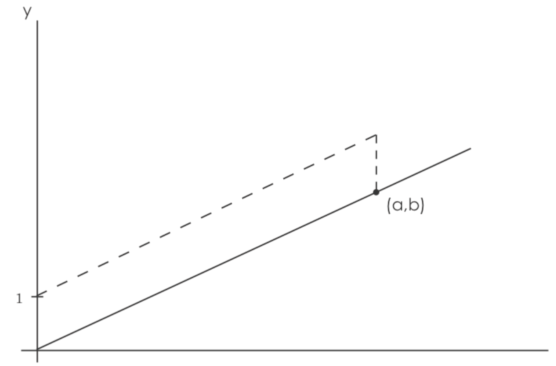

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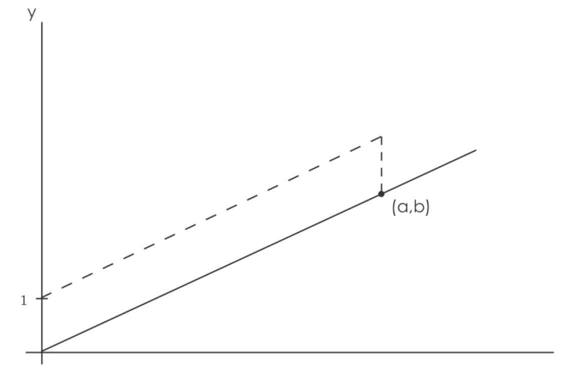
$$\begin{aligned} \sigma_{\mathcal{K}_1}(u,v) &:= \sum_{(m,n) \in \mathcal{K}_1 \cap \mathbb{Z}^2} u^m v^n = \sigma_{\Pi_1}(u,v) \left(\sum_{j \geq 0} v^j \right) \left(\sum_{k \geq 0} u^{ka} v^{kb} \right) \\ &= \frac{\sigma_{\Pi_1}(u,v)}{(1-v)\left(1-u^a v^b\right)} \, , \end{aligned}$$

where Π_1 is the fundamental parallelogram of $\mathcal{K}_1$:

$$\Pi_1 = \{\lambda_1(0,1) + \lambda_2(a,b) : \, 0 \leq \lambda_1, \lambda_2 < 1\} \, .$$

Carlitz Reciprocity



The integer points in this parallelogram are

$$\Pi_1 \cap \mathbb{Z}^2 = \left\{ (0,0), \left(k, \left\lfloor \frac{kb}{a} \right\rfloor + 1 \right) : \, 1 \le k \le a-1, \, k \in \mathbb{Z} \right\}.$$

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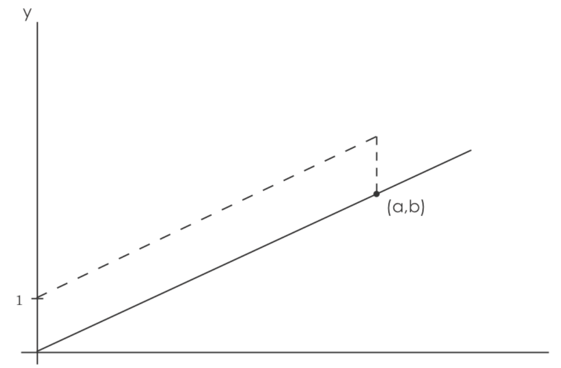

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from which we obtain

$$\sigma_{K_1}(u,v) = \frac{1 + \sum_{k=1}^{a-1} u^k v^{\left\lfloor \frac{kb}{a} \right\rfloor + 1}}{(1-v)(1-u^a v^b)} = \frac{1 + uv\, \mathrm{c}\,(v,u;b,a)}{(v-1)\,(u^a v^b - 1)}.$$

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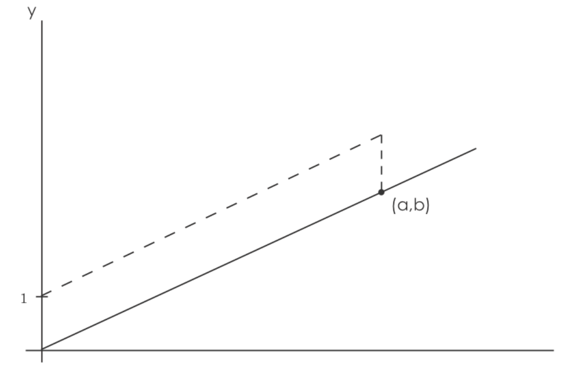

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Analogously, one computes $\sigma_{K_2}(u,v) = \frac{u + uv\,\mathrm{c}(u,v;a,b)}{(u-1)\left(u^a v^b - 1\right)}$ and Carlitz's reciprocity law follows from

$$\sigma_{K_1}(u,v) + \sigma_{K_2}(u,v) = \sigma_{\mathbb{R}^2_{\ge 0}}(u,v) = \frac{1}{(1-u)(1-v)} .$$

Higher Dimensions

Our proof has a natural generalization to the higher-dimensional Dedekind–Carlitz polynomials

$$\mathrm{c}\left(u_1, u_2, \ldots, u_n; a_1, a_2, \ldots, a_n\right) := \sum_{k=1}^{a_n-1} u_1^{\left\lfloor \frac{ka_1}{a_n} \right\rfloor} u_2^{\left\lfloor \frac{ka_2}{a_n} \right\rfloor} \cdots u_{n-1}^{\left\lfloor \frac{ka_{n-1}}{a_n} \right\rfloor} u_n^{k-1},$$

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where $u_1, u_2, \dots, u_n$ are indeterminates and $a_1, a_2, \dots, a_n$ are positive integers. Berndt–Dieter proved that if $a_1, a_2, \dots, a_n$ are pairwise relatively prime then

$$\begin{aligned}
&(u_n - 1)\, \mathrm{c}\left(u_1, u_2, \dots, u_n; a_1, a_2, \dots, a_n\right) \\
&\qquad + (u_{n-1} - 1)\, \mathrm{c}\left(u_n, u_1, \dots, u_{n-2}, u_{n-1}; a_n, a_1, \dots, a_{n-2}, a_{n-1}\right) \\
&\qquad + \cdots + (u_1 - 1)\, \mathrm{c}\left(u_2, u_3, \dots, u_n, u_1; a_2, a_3, \dots, a_n, a_1\right) \\
&= u_1^{a_1 - 1} u_2^{a_2 - 1} \cdots u_n^{a_n - 1} - 1\,.
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&= u_1^{a_1-1} u_2^{a_2-1} \cdots u_n^{a_n-1} - 1\,.
\end{aligned}$$

We could shift the cones involved in our proofs by a fixed vector. This gives rise to shifts in the greatest-integer functions, and the resulting Carlitz sums are polynomial analogues of Dedekind–Rademacher sums.

Computational Complexity

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Fortunately, Barvinok proved in the 1990's that in fixed dimension, the integer-point transform $\sigma_{\mathcal{P}}(z_1, z_2, \ldots, z_d)$ of a rational polyhedron $\mathcal{P}$ can be computed as a sum of rational functions in $z_1, z_2, \ldots, z_d$ in time polynomial in the input size of $\mathcal{P}$.

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Theorem For fixed n, the higher-dimensional Dedekind–Carlitz polynomial $\mathrm{c}\left(u_1, u_2, \ldots, u_n; a_1, a_2, \ldots, a_n\right)$ can be computed in time polynomial in the size of $a_1, a_2, \ldots, a_n$.

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In particular, there is a more economical way to write the "long" polynomial $\mathrm{c}(u_1, u_2, \ldots, u_n; a_1, a_2, \ldots, a_n)$ as a short sum of rational functions. Our theorem also implies that any Dedekind-like sum that can be derived from Dedekind–Carlitz polynomials can also be computed efficiently.

General 2-Dimensional Rational Cones

Theorem Let $a, b, c, d \in \mathbb{Z}_{>0}$ such that $ad > bc$ and $\gcd(a,b) = \gcd(c,d) = 1$, and define $x, y \in \mathbb{Z}$ through $ax + by = 1$. Then the cone $\mathcal{K} := \{\lambda(a,b) + \mu(c,d) : \lambda, \mu \geq 0\}$ has the integer-point transform

$$\sigma_{\mathcal{K}}(u,v) = \frac{1 + u^{a-y} v^{b+x} \, \mathrm{c}\left(u^a v^b, u^{-y} v^x; cx + dy, ad - bc\right)}{\left(u^a v^b - 1\right)\left(u^c v^d - 1\right)} .$$

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We can decompose $\mathbb{R}^2_{\geq 0}$ into $\mathcal{K}$ plus two more cones whose integer-point transform can be computed as shown earlier. This immediately yields a polynomial generalization of a three-term reciprocity law of Pommersheim:

Theorem Let a, b, c, d, x, y be as above, then

$$\begin{aligned} uv(u-1)\left(u^a v^b - 1\right) \mathrm{c}\,(v, u; d, c) + uv(v-1)\left(u^c v^d - 1\right) \mathrm{c}\,(u, v; a, b) \\ + u^{a-y} v^{b+x} (u-1)(v-1)\, \mathrm{c}\left(u^a v^b, u^{-y} v^x; cx + dy, ad - bc\right) \\ = u^{a+c} v^{b+d} - u^a v^b (uv - v + 1) - u^c v^d (uv - u + 1) + uv \, . \end{aligned}$$

Enter Brion Decompositions

Brion's theorem says that for a rational convex polytope $\mathcal{P}$, we have the following identity of rational functions:

$$\sigma_{\mathcal{P}}(\boldsymbol{z}) = \sum_{\boldsymbol{v}} \sigma_{\mathcal{K}_{\boldsymbol{v}}}(\boldsymbol{z}) \, ,$$

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Let a and b be relatively prime positive integers and Δ the triangle with vertices $(0,0)$, $(a,0)$, and $(0,b)$. Brion's theorem allows us to give a novel expression for the Dedekind–Carlitz polynomial as the integer-point transform of a certain triangle.

Theorem $\quad (u-1)\, \sigma_{\Delta}(u,v) = u^a v \, \mathrm{c}\left(\frac{1}{u}, v; a, b\right) + u\left(u^a + v^b\right) - \dfrac{v^{b+1}-1}{v-1} \, .$

Mordell–Pommersheim Tetrahedra

Mordell established the first connection between lattice point formulas and Dedekind sums in the 1950's; his theorem below was vastly generalized in the 1990's by Pommersheim.

Let $\mathcal{T}$ be the convex hull of $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, and $(0, 0, 0)$, where a, b, and c are pairwise relatively prime positive integers. Then the Ehrhart polynomial $\#\left(t\mathcal{T} \cap \mathbb{Z}^3\right)$ of $\mathcal{T}$ is

$$\begin{aligned} L_{\mathcal{T}}(t) \quad = \quad & \frac{abc}{6}t^3 + \frac{ab+ac+bc+1}{4}t^2 \\ & + \left(\frac{3}{4} + \frac{a+b+c}{4} + \frac{1}{12}\left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} + \frac{1}{abc}\right)\right. \\ & \left. - \mathrm{s}\,(bc, a) - \mathrm{s}\,(ca, b) - \mathrm{s}\,(ab, c)\right)t + 1\,. \end{aligned}$$

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For positive integers a, b, c , and indeterminates u, v, w , we define the Dedekind–Rademacher–Carlitz sum

$$\operatorname{drc}(u, v, w; a, b, c) := \sum_{k=0}^{c-1} \sum_{j=0}^{b-1} u^{\left\lfloor \frac{ja}{b} + \frac{ka}{c} \right\rfloor} v^j w^k .$$

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Theorem Let $\mathcal{T}$ be the convex hull of $(a,0,0)$, $(0,b,0)$, $(0,0,c)$, and $(0,0,0)$ where a, b, and c are pairwise relatively prime positive integers. Then

$$\begin{aligned}
&(u-1)(v-1)(w-1)\left(u^a-v^b\right)\left(u^a-w^c\right)\left(v^b-w^c\right)\sigma_{t\mathcal{T}}(u,v,w)\\
&\quad = u^{(t+2)a}(v-1)(w-1)\left(v^b-w^c\right)\left((u-1)+\operatorname{drc}\left(u^{-1},v,w;a,b,c\right)\right)\\
&\qquad -v^{(t+2)b}(u-1)(w-1)\left(u^a-w^c\right)\left((v-1)+\operatorname{drc}\left(v^{-1},u,w;b,a,c\right)\right)\\
&\qquad +w^{(t+2)c}(u-1)(w-1)\left(u^a-v^b\right)\left((w-1)+\operatorname{drc}\left(w^{-1},u,v;c,a,b\right)\right)\\
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\end{aligned}$$

The Mordell–Pommersheim theorem follows with $L_{\mathcal{T}}(t) = \sigma_{t\mathcal{T}}(1, 1, 1)$.