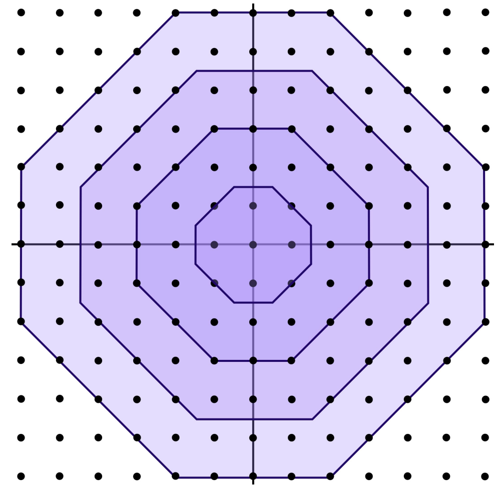
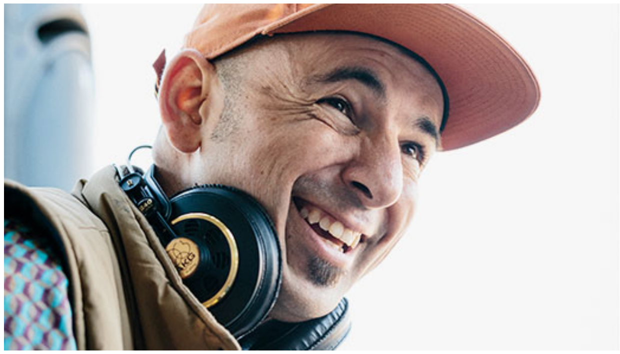
The Arithmetic of Coxeter Permutahedra





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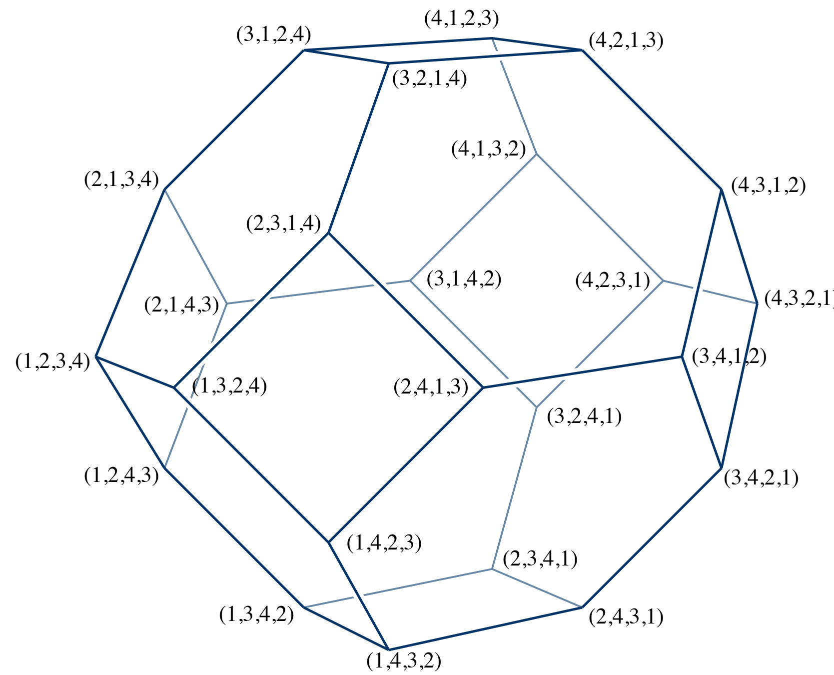
The Menu

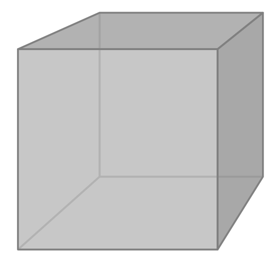
- Ehrhart (quasi-)polynomials
- Zonotopes
- Coxeter permutahedra
- Signed graphs
- Tree generating functions



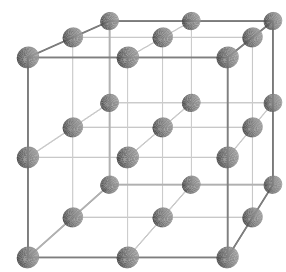
Measuring Polytopes

Rational polytope — convex hull of finitely many points in $\mathbb{Q}^d$
— solution set of a system of linear (in-)equalities with integer coefficients

Goal: measuring...



- volume $\mathrm{vol}(\mathcal{P}) = \lim_{t \to \infty} \frac{1}{t^d} \left| \mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d \right|$



- discrete volume $\left| \mathcal{P} \cap \mathbb{Z}^d \right|$

Ehrhart function $\mathrm{ehr}_{\mathcal{P}}(t) := \left| \mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d \right| = \left| t\mathcal{P} \cap \mathbb{Z}^d \right|$ for $t \in \mathbb{Z}_{>0}$

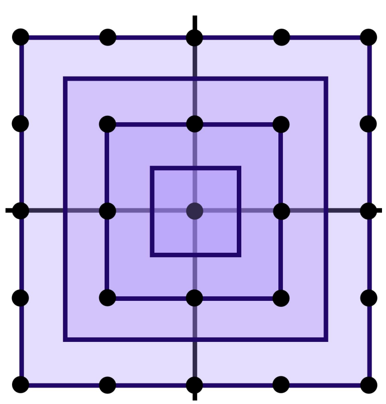
Discrete Volumes & Ehrhart Quasipolynomials

Rational polytope — convex hull of finitely many points in $\mathbb{Q}^d$

$q(t) = c_d(t)\, t^d + \cdots + c_0(t)$ is a quasipolynomial if $c_0(t), \ldots, c_d(t)$ are periodic functions; the lcm of their periods is the period of $q(t)$.

Theorem (Ehrhart 1962) For any rational polytope $\mathcal{P} \subset \mathbb{R}^d$, $\operatorname{ehr}_{\mathcal{P}}(t) := \left| t\mathcal{P} \cap \mathbb{Z}^d \right|$ is a quasipolynomial in t whose period divides the lcm of the denominators of the vertex coordinates of P.

Example $\mathcal{P} = [-\frac{1}{2}, \frac{1}{2}]^2$



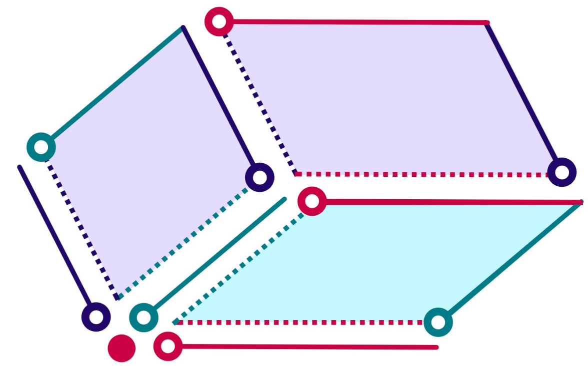
Why care about... Ehrhart (Quasi-)Polynomials

- Linear systems are everywhere, and so polyhedra are everywhere.
- In applications, the volume of the polytope represented by a linear system measures some fundamental data of this system ("average").
- Polytopes are basic geometric objects, yet even for these basic objects volume computation is hard and there remain many open problems.
- Many discrete problems in various mathematical areas are linear problems, thus they ask for the discrete volume of a polytope in disguise.
- Much discrete geometry can be modeled using polynomials and, conversely, many combinatorial polynomials can be modeled geometrically.

Zonotopes

Zonotope — Minkowski sum of line segments $\mathcal{Z} = \sum_{j=1}^{n} [\mathbf{a}_j, \mathbf{b}_j]$

Shephard (1974) Decomposition of $\mathcal{Z}$ into translates of half-open parallelepipeds spanned by the linearly independent subsets of $\{\mathbf{b}_j - \mathbf{a}_j : 1 \le j \le n\}$.



Stanley (1991) For a finite set of vectors $\mathbf{U} \subset \mathbb{Z}^d$, let $\mathcal{Z}(\mathbf{U}) := \sum_{\mathbf{u} \in \mathbf{U}} [\mathbf{0}, \mathbf{u}]$
Then

$$\mathrm{ehr}_{\mathcal{Z}(\mathbf{U})}(t) = \sum_{\substack{\mathbf{W} \subseteq \mathbf{U} \\ \text{lin. indep.}}} \mathrm{vol}(\mathbf{W})\, t^{|\mathbf{W}|}$$

where $|\mathbf{W}|$ denotes the number of vectors in $\mathbf{W}$ and $\mathrm{vol}(\mathbf{W})$ is the relative volume of the parallelepiped generated by $\mathbf{W}$.

Lie Combinatorics

Finite crystallographic root systems

$$\begin{aligned} A_{n-1} &:= \{\pm(\mathbf{e}_i - \mathbf{e}_j) : 1 \le i < j \le n\} \\ B_n &:= \{\pm(\mathbf{e}_i - \mathbf{e}_j), \pm(\mathbf{e}_i + \mathbf{e}_j) : 1 \le i < j \le n\} \cup \{\pm\mathbf{e}_i : 1 \le i \le n\} \\ C_n &:= \{\pm(\mathbf{e}_i - \mathbf{e}_j), \pm(\mathbf{e}_i + \mathbf{e}_j) : 1 \le i < j \le n\} \cup \{\pm 2\,\mathbf{e}_i : 1 \le i \le n\} \\ D_n &:= \{\pm(\mathbf{e}_i - \mathbf{e}_j), \pm(\mathbf{e}_i + \mathbf{e}_j) : 1 \le i < j \le n\} \end{aligned}$$

$\ldots$ and E_6, E_7, E_8, F_4, G_2 .

Positive roots are obtained by choosing the plus sign in each $\pm$ above.

Standard Coxeter permutahedron of the finite root system Φ

$$\Pi(\Phi) := \sum_{\alpha \in \Phi^+} \left[-\tfrac{\alpha}{2}, \tfrac{\alpha}{2}\right] = \operatorname{conv}\{w \cdot \rho : w \in W\}$$

where $\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ and W is the Weyl group of Φ

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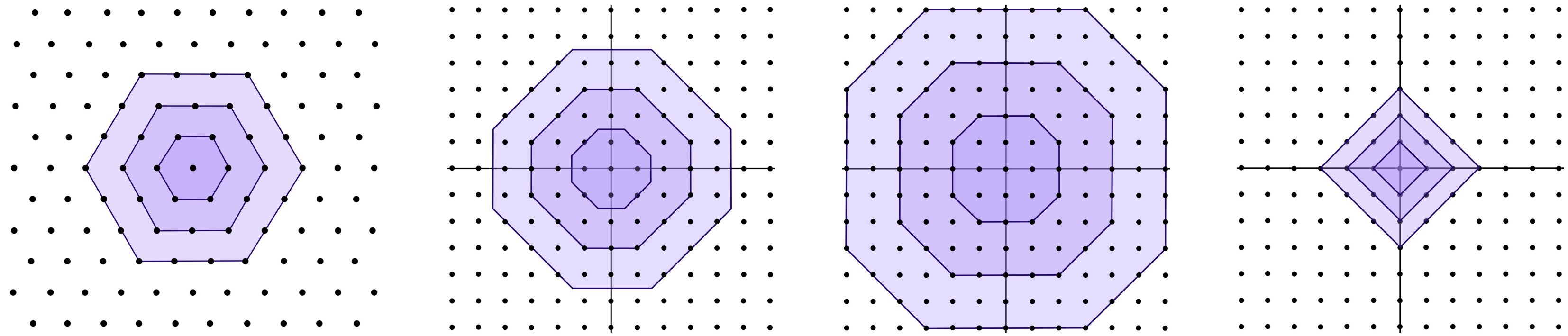
Standard Coxeter permutahedron of the finite root system Φ

$$\Pi(\Phi) \ := \sum_{\alpha \in \Phi^+} \left[-\tfrac{\alpha}{2}, \tfrac{\alpha}{2}\right]$$

Integral Coxeter permutahedron $\Pi^{\mathbb{Z}}(\Phi) \ := \displaystyle\sum_{\alpha \in \Phi^+} [0, \alpha]$

Standard Coxeter Permutahedra



$$\begin{aligned}
A_{n-1} &= \{\pm(\mathbf{e}_i - \mathbf{e}_j) : 1 \le i < j \le n\} \\
B_n &= \{\pm(\mathbf{e}_i - \mathbf{e}_j), \pm(\mathbf{e}_i + \mathbf{e}_j) : 1 \le i < j \le n\} \cup \{\pm\mathbf{e}_i : 1 \le i \le n\} \\
C_n &= \{\pm(\mathbf{e}_i - \mathbf{e}_j), \pm(\mathbf{e}_i + \mathbf{e}_j) : 1 \le i < j \le n\} \cup \{\pm 2\,\mathbf{e}_i : 1 \le i \le n\} \\
D_n &= \{\pm(\mathbf{e}_i - \mathbf{e}_j), \pm(\mathbf{e}_i + \mathbf{e}_j) : 1 \le i < j \le n\} \\
\Pi(A_{n-1}) &= \text{conv}\{\text{permutations of } \tfrac{1}{2}(-n+1, -n+3, \ldots, n-3, n-1)\} \\
\Pi(B_n) &= \text{conv}\{\text{signed permutations of } \tfrac{1}{2}(1, 3, \ldots, 2n-1)\} \\
\Pi(C_n) &= \text{conv}\{\text{signed permutations of } (1, 2, \ldots, n)\} \\
\Pi(D_n) &= \text{conv}\{\text{evenly signed permutations of } (0, 1, \ldots, n-1)\}
\end{aligned}$$

Why care about... Coxeter Permutahedra

- Many questions about permutations can be answered looking at the geometry of the permutahedron
- Fundamental objects in the representation theory of semisimple Lie algebras
- Connections to optimization (Ardila–Castillo–Eur–Postnikov 2020)
- Zonotopes with natural connections to tree enumeration

Signed Graphs

A signed graph $G = (V, E, \sigma)$ comes with a signature $\sigma : E_* \to \{\pm\}$

A simple cycle is balanced if its product of signs is $+$. A signed graph is balanced if it contains no half edges and all of its simple cycles are balanced.

An all-negative signed graph is balanced if and only if it is bipartite.

A signed graph is balanced if and only if it has no half edges and can be switched to an all-positive signed graph.

Signed Graphs and Root Systems

Zaslavsky Encoding (1981) of a subset $S \subseteq \Phi^+$ into the signed graph G_S with

- a positive edge ij for each $\mathbf{e}_i - \mathbf{e}_j \in S$
- a negative edge ij for each $\mathbf{e}_i + \mathbf{e}_j \in S$
- a halfedge at j for each $\mathbf{e}_j \in S$
- a negative loop at j for each $2\mathbf{e}_j \in S$

Linear independent subsets of Φ^+ correspond precisely to signed pseudo-forests which consist of signed trees plus possibly

- a single halfedge (halfedge-tree)
- a single loop (loop-tree)
- a single unbalanced cycle (pseudotree)

$$|\Phi_G| = n - \mathrm{tc}(G)$$

$$\mathrm{vol}(\Phi_G) = 2^{\mathrm{pc}(G)+\mathrm{lc}(G)}$$

Why care about... Signed Graphs

- Earliest appearance in social psychology (Heider 1946, Cartwright–Harary 1956) "The enemy of my enemy is my friend"
- Type-B analogues of graphs, natural from the viewpoint of incidence matrices
- Applications to
  - Knot theory (positive/negative crossings)
  - Biology (perturbed large-scale biological networks
  - Chemistry (Möbius systems)
  - Physics (spin glasses—mixed Ising model)
  - Computer science (correlation clustering)

Integral Coxeter Permutahedra

Fix $\Phi \in \{A_n, B_n, C_n, D_n : n \geq 2\}$ and consider $\Pi^{\mathbb{Z}}(\Phi) = \sum_{\alpha \in \Phi^+} [0, \alpha]$

Linear independent subsets of Φ^+ correspond precisely to signed pseudo-forests which consist of signed trees plus possibly

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$$|\Phi_G| = n - \mathrm{tc}(G)$$

$$\mathrm{vol}(\Phi_G) = 2^{\mathrm{pc}(G)+\mathrm{lc}(G)}$$

Ardila–Castillo–Henley (2015) Let $\mathcal{F}(\Phi)$ be the set of Φ-forests. Then

$$\mathrm{ehr}_{\Pi^{\mathbb{Z}}(\Phi)}(t) = \sum_{G \in \mathcal{F}(\Phi)} 2^{\mathrm{pc}(G)+\mathrm{lc}(G)}\, t^{n-\mathrm{tc}(G)}.$$

Almost Integral Zonotopes

Lemma Let $\mathbf{U} \subset \mathbb{Z}^d$ be a finite set and $\mathbf{v} \in \mathbb{Q}^d$. Then

$$\operatorname{ehr}_{\mathbf{v}+\mathcal{Z}(\mathbf{U})}(t) \;=\; \sum_{\substack{\mathbf{W}\subseteq\mathbf{U} \\ \text{lin. indep.}}} \chi_{\mathbf{W}}(t)\,\operatorname{vol}(\mathbf{W})\,t^{|\mathbf{W}|}$$

where $\chi_{\mathbf{W}}(t) := \begin{cases} 1 & \text{if } (t\mathbf{v} + \operatorname{span}(\mathbf{W})) \cap \mathbb{Z}^d \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$

Ardlia–MB–McWhirter Fix $\Phi \in \{A_n : n \geq 2 \text{ even}\} \cup \{B_n : n \geq 1\}$. Let $\mathcal{F}(\Phi)$ be the set of Φ-forests and $\mathcal{E}(\Phi) \subseteq \mathcal{F}(\Phi)$ be the set of Φ-forests such that every tree component has an even number of vertices. Then

$$\operatorname{ehr}_{\Pi(\Phi)}(t) \;=\; \begin{cases} \displaystyle\sum_{G\in\mathcal{F}(\Phi)} 2^{\operatorname{pc}(G)} t^{n-\operatorname{tc}(G)} & \text{if } t \text{ is even,} \\ \displaystyle\sum_{G\in\mathcal{E}(\Phi)} 2^{\operatorname{pc}(G)} t^{n-\operatorname{tc}(G)} & \text{if } t \text{ is odd.} \end{cases}$$

Exponential Generating Functions

Lambert W-function $\qquad W(x) = \sum_{n \geq 1} (-n)^{n-1} \frac{x^n}{n!} \qquad W(x)\, e^{W(x)} = x$

There are $t_n := n^{n-2}$ trees on $[n]$, with exponential generation function

$$\sum_{n \geq 1} t_n \frac{x^n}{n!} \; = \; -W(-x) - \frac{1}{2} W(-x)^2$$

Sample tree generating function magic

$$\sum_{n \geq 0} \mathrm{ehr}_{\Pi^{\mathbb{Z}}(A_{n-1})}(t) \frac{x^n}{n!} \; = \; \sum_{n \geq 0} \sum_{\substack{\text{forests} \\ G \text{ on } [n]}} t^{n - \mathrm{tc}(G)} \frac{x^n}{n!}$$

Exponential Generating Functions

Ardlia–MB–McWhirter Exponential generating functions for integral and standard Coxeter permutahedra, e.g., for t odd,

$$\sum_{n\ge 0} \mathrm{ehr}_{\Pi(A_{2n-1})}(t)\,\frac{x^{2n}}{(2n)!} \quad = \quad \exp\left(-\frac{W(-tx)+W(tx)}{2t} - \frac{W(-tx)^2+W(tx)^2}{4t}\right)$$

$$\sum_{n\ge 0} \mathrm{ehr}_{\Pi(B_n)}(t)\,\frac{x^{n}}{n!} \quad = \quad \frac{\exp\left(-\frac{W(-2tx)+W(2tx)}{4t} - \frac{W(-2tx)^2+W(2tx)^2}{8t}\right)}{\sqrt{1+W(-2tx)}}$$