$\binom{-5}{12}$ and Other Combinatorial Reciprocity Instances

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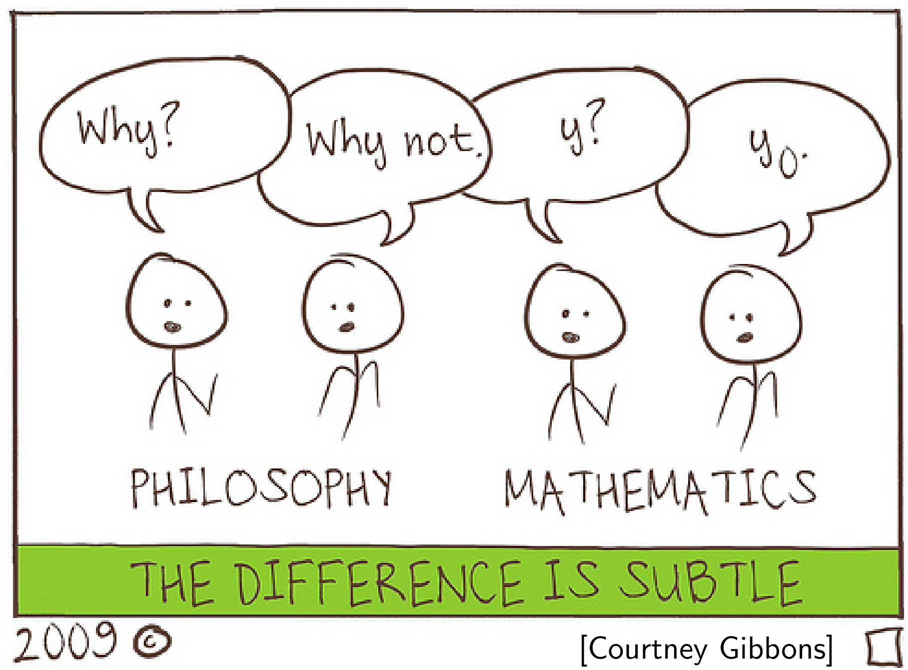

[Courtney Gibbons]

Act 1: Binomial Coefficients

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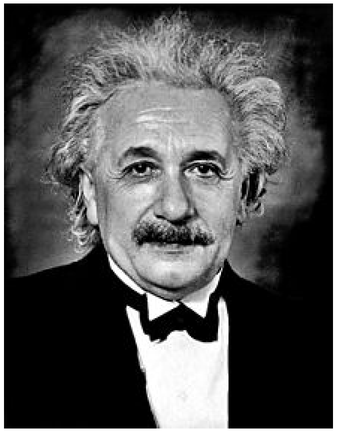

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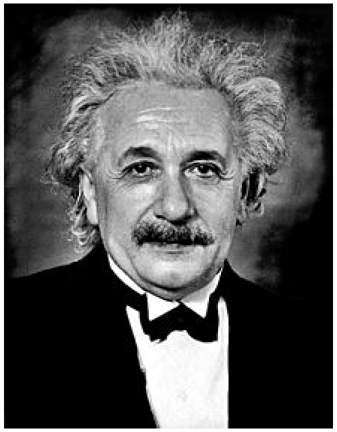

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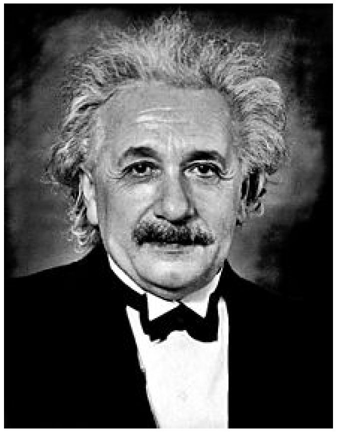

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$$(-1)^k \binom{-n}{k} = \binom{n+k-1}{k}$$

Act 2: Chromatic Polynomials of Graphs

Proper n-coloring of G — labeling of the nodes of G with $1, 2, \ldots, n$ such that adjacent nodes get different labels

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Theorem (Birkhoff 1912, Whitney 1932) $\chi_G(n)$ is a polynomial in n.

Proof Let c_k be the number of ways of breaking up the nodes V into k monochromatic subsets. Then

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Theorem (Stanley 1973) $(-1)^{|V|}\chi_G(-1)$ equals the number of acyclic orientations of G. More generally, $(-1)^{|V|}\chi_G(-n)$ equals the number of pairs (acyclic orientation α of G, compatible n-coloring).

$$\alpha = \frac{1}{1(p-1)}$$

$$\beta = \frac{p+1}{1.2\,(p-1)^2}$$

$$\gamma = \frac{pp+4p+1}{1.2.3\,(p-1)^3}$$

$$\delta = \frac{p^3+11p^2+11p+1}{1.2.3.4\,(p-1)^4}$$

$$\varepsilon = \frac{p^4+26p^3+66p^2+26p+1}{1.2.3.4.5\,(p-1)^5}$$

$$\zeta = \frac{p^5+57p^4+302p^3+302p^2+57p+1}{1.2.3.4.5.6\,(p-1)^6}$$

$$\eta = \frac{p^6+120p^5+1191p^4+2416p^3+1191p^2+120p+1}{1.2.3.4.5.6.7\,(p-1)^7}$$

&c.

L. Euler, 1755.

Eulerian Polynomials

$$\frac{A_n(p)/p}{n!(p-1)^n} \qquad (1 \le n \le 7)$$

Act 3: Eulerian Polynomials

$\left\langle {n \atop k} \right\rangle$ — number of permutations of $\{1, 2, \ldots, n\}$ with exactly k descents

Exercise 1 Show that $\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-k-1} \right\rangle$

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Let $E_n(x) := \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k$, the n^{th} Eulerian polynomial. Exercise 1 says

$$x^{n-1} \, E_n\left(\frac{1}{x}\right) = E_n(x)$$

Exercise 2 Show that $\sum_{t=0}^{\infty} t^{n-1} \, x^t = \frac{E_n(x)}{(1-x)^n}$

Exercise 3 Re-prove Exercise 1 via Exercise 2.

Act 4: Pick's Theorem

For a lattice polygon $\mathcal{P}$ containing I interior and B boundary lattice point, Pick's Theorem (1899) tells us how to compute the area of P:

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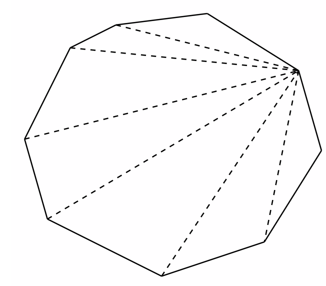
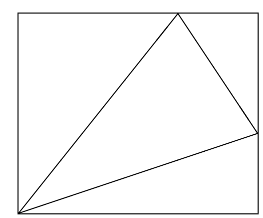
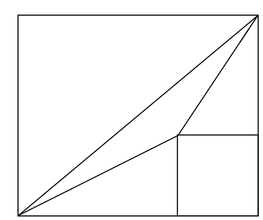
Do-it-yourself proof:



(1) Convince yourself that Pick's formula is "additive".



(2) Reduce to rectangles and right-angled triangles.



(3) Prove Pick's formula for these two cases.

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For $k \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(k) := \# \left(k\mathcal{P} \cap \mathbb{Z}^2 \right)$

$$L_{\mathcal{P}}(k) \ = \ A\,k^2 + \frac{1}{2}B\,k + 1$$

$$L_{\mathcal{P}^\circ}(k) \ = \ A\,k^2 - \frac{1}{2}B\,k + 1$$

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Example Triangle Δ with vertices $(0,0)$, $(1,0)$, and $(0,1)$

$$L_\Delta(k) \ = \ \binom{k+2}{2} \qquad\qquad L_{\Delta^\circ}(k) \ = \ \binom{k-1}{2}$$

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Theorem (Ehrhart 1962, Macdonald 1971) If $\mathcal{P}$ is a d-dimensional lattice polytope, then $L_{\mathcal{P}}(k)$ is a polynomial in k and $(-1)^d L_{\mathcal{P}}(-k) \ = \ L_{\mathcal{P}^\circ}(k)$