Weighted lattice point sums in lattice polytopes

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The Bott–Brion–Dehn–Ehrhart–Euler–Khovanskii–Maclaurin–Pukhlikov–Sommerville–Vergne formula for simple lattice polytopes

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The Menu

- Lattice-point counting in lattice polytopes: (weighted) Ehrhart polynomials and their reciprocity

- Face-counting for simple polytopes: (generalized) Dehn–Sommerville relations

Our goal Give a unifying reciprocity theorem

Secondary goal Entice (some of) you to study weighted Ehrhart polynomials

Ehrhart–Macdonald Reciprocity

V — real vector space of dimension n equipped with a lattice $M \subset V$

$P \subset V$ — (n-dimensional) lattice polytope (i.e., vertices in M)

For $t \in \mathbb{Z}_{>0}$ let $E_P(t) := |M \cap tP|$

Ehrhart–Macdonald (1960s) $E_P(t)$ is a polynomial in t (of degree $\dim(P)$ and with constant term 1) that satisfies

$$E_P(-t) \ = \ (-1)^{\dim(P)} E_{P^\circ}(t) \, .$$

Example $P = \operatorname{conv}\{(\pm 1, \pm 1, 1), (0,0,1)\}$

$$E_P(t) \ = \ \tfrac{4}{3} \, t^3 + 4 \, t^2 + \tfrac{11}{3} \, t + 1$$

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- In the dictionary $P \longleftrightarrow$ toric variety (if P is a very ample), $E_P(t)$ equals the Hilbert polynomial of this toric variety under the projective embedding given by the very ample divisor associated with P.
- Ehrhart–Macdonald is part of an illustrious series of combinatorial reciprocity theorems

Ehrhart Polynomials

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Natural, currently *en vogue* questions:

- (Sub-)Classification of Ehrhart polynomials
- Families of polytopes with positive/unimodal/... Ehrhart coefficients

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For a homogeneous polynomial φ let $E_{\varphi,P}(t) \ := \sum_{m \in M \cap tP} \varphi(m)$

Brion–Vergne (1997) $E_{\varphi,P}(t)$ is a polynomial in t (of degree $\dim(P) + \deg(\varphi)$ and with constant term $\varphi(0)$) that satisfies

$$E_{\varphi,P}(-t) \ = \ (-1)^{\dim(P)+\deg(\varphi)} E_{\varphi,P^\circ}(t) \, .$$

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Possible (and possibly *en vogue*) questions:

- Structural theorems (*à la* $h_P^* \geq 0$) under certain conditions
- Families of polytopes with positive/unimodal/... Ehrhart coefficients
- Special cases, e.g., $\varphi(m) = m_1$ or $\varphi(m) = m_1 + m_2 + \cdots + m_n$

Dehn–Sommerville Relations

P is simple if each vertex meets n edges

$\mathcal{F}$ — set of faces of P

The h-polynomial of P is $h_P(y) := \sum_{F \in \mathcal{F}} (y-1)^{\dim(F)}$

Dehn–Sommerville (early 1900s) If P is simple then $y^n h_P(\frac{1}{y}) = h_P(y)$.

Example If P is a simplex then $h_P(y) = y^n + y^{n-1} + \cdots + 1$

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- In the dictionary $P \longleftrightarrow$ toric variety, Dehn–Sommerville corresponds to Poincaré duality for the rational cohomology of the toric variety attached to P.
- Combinatorially, Dehn–Sommerville follows from the fact that the face lattice of a polytope is Eulerian and thus its zeta polynomial is even/odd.

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Natural, equally *en vogue* questions:

- (Sub-)Classification of h-polynomials
- Extensions to simplicial/polyhedral/... complexes

Main Theorem (1st Version)

$$E_{\varphi,P}(t) := \sum_{m \in M \cap tP} \varphi(m) \qquad E_{\varphi,P}(-t) \ = \ (-1)^{\dim(P)+\deg(\varphi)} E_{\varphi,P^\circ}(t)$$

$$h_P(y) := \sum_{F \in \mathcal{F}} (y-1)^{\dim(F)} \qquad y^n h_P(\tfrac{1}{y}) = h_P(y)$$

Let $G_{\varphi,P}(t,y) \ := \ (y+1)^{\deg(\varphi)} \sum_{F \in \mathcal{F}} (y+1)^{\dim(F)} (-y)^{\operatorname{codim}(F)} E_{\varphi,F}(t)$

Theorem (MB–Gunnels–Materov) If P is a simple lattice polytope then

$$G_{\varphi,P}(t,y) \ = \ (-y)^{\dim(P)+\deg(\varphi)} \, G_{\varphi,P}(-t, \tfrac{1}{y}) \, .$$

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If $\varphi = 1$ then $E_{\varphi,F}(t) = E_F(t)$ and the constant terms (in y) of

$$\begin{aligned} (-1)^{\dim(P)} \, G_{\varphi=1,P}(-t,y) \ &= \ \sum_{F \in \mathcal{F}} (y+1)^{\dim(F)} (-y)^{\operatorname{codim}(F)} E_F(-t) \\ y^{\dim(P)} \, G_{\varphi=1,P}(t,\tfrac{1}{y}) \ &= \ \sum_{F \in \mathcal{F}} (y+1)^{\dim(F)} (-1)^{\operatorname{codim}(F)} E_F(t) \end{aligned}$$

are $\sum_{F \in \mathcal{F}} E_{F^\circ}(t) = E_P(t)$ and $\sum_{F \in \mathcal{F}} (-1)^{\operatorname{codim}(F)} E_F(t) = E_{P^\circ}(t)$

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If $\varphi = 1$ and $t = 0$ then

$$G_{\varphi=1,P}(0,y) \ = \ \sum_{F \in \mathcal{F}} (y+1)^{\dim(F)} (-y)^{\operatorname{codim}(F)} \ = \ (-y)^{\dim(P)} h_P(-\tfrac{1}{y})$$

and

$$(-y)^{\dim(P)}\, G_{\varphi,P}(0, \tfrac{1}{y}) \ = \ \sum_{F \in \mathcal{F}} (-1-y)^{\dim(F)} \ = \ h_P(-y)$$

g-Polynomials

We define polynomials $f_P(x)$ and $g_P(x)$ recursively by dimension:

- $f_\varnothing(x) = g_\varnothing(x) = 1$
- $$f_P(x) = \sum_{F \in \mathcal{F} \setminus \{P\}} g_F(x)(x-1)^{n-\dim(F)-1} = \sum_{j=0}^{\dim(P)} f_j x^j$$

  $$g_P(x) = f_0 + (f_1 - f_0)x + (f_2 - f_1)x^2 + \cdots + (f_m - f_{m-1})x^m$$

  where $m = \lfloor \frac{\dim(P)}{2} \rfloor$

Master Duality Theorem (Stanley 1974) $f_P(x) \ = \ x^{\dim(P)} f_P(\frac{1}{x})$

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- This definition of $f_P(x)$ is dual to that of the h-polynomial. It favors simplicial polytopes, in that Dehn–Sommerville holds with no $g_P(x)$ corrections.
- In the dictionary $P \longleftrightarrow$ toric variety, $g_P(x)$ takes into account the intersection cohomology of the variety.

Main Theorem (2nd Version)

Let $\widetilde{F}$ be the dual face of F in the polar polytope of P and $\widetilde{g}_F(x) := g_{\widetilde{F}}(x)$

$$G_{\varphi,P}(t,y) \;:=\; (y+1)^{\deg(\varphi)} \sum_{F \in \mathcal{F}} (y+1)^{\dim(F)} (-y)^{\operatorname{codim}(F)} E_{\varphi,F}(t)\, \widetilde{g}_F(-\tfrac{1}{y})$$

Remark If P is simple then $\widetilde{F}$ is a simplex for every proper face F and thus $\widetilde{g}_F(x) = 1$, recovering our earlier definition.

Theorem (MB–Gunnels–Materov) If P is a lattice polytope then

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Example $P = \operatorname{conv}\{(\pm 1, \pm 1, 1), (0,0,1)\}$

$$G_{\varphi,P}(t,y) \ = \ \left(\tfrac{4}{3}t^3 - 4t^2 + \tfrac{11}{3}t - 1\right) y^3 + \left(4t^3 - 4t^2 - t + 2\right) y^2$$
$$+ \left(4t^3 + 4t^2 - t - 2\right) y + \left(\tfrac{4}{3}t^3 + 4t^2 + \tfrac{11}{3}t + 1\right)$$

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Ingredients Brion–Vergne reciprocity and

$$x^{\dim(P)+1} g_P(\tfrac{1}{x}) \ = \ \sum_{F \in \mathcal{F}} g_F(x)(x-1)^{n-\dim(F)}$$

Euler–Maclaurin Summation

We perturb a given polytope

$$\begin{aligned} P &= \{x \in V : \langle x, u_F \rangle + \lambda_F \geq 0 \text{ for each facet } F\} \\ P_{t,y}(h) &:= \{x \in V : \langle x, u_F \rangle + t(y+1)\lambda_F + h_F \geq 0 \text{ for each facet } F\} \end{aligned}$$

using a vector $h = (h_F : F \text{ facet of } P)$

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Euler–Maclaurin $\quad \mathrm{Todd}(\frac{\partial}{\partial h}) := \dfrac{\frac{\partial}{\partial h}}{1 - e^{-\frac{\partial}{\partial h}}} = \displaystyle\sum_{k \geq 0} (-1)^k \frac{B_k}{k!} \left(\frac{\partial}{\partial h}\right)^k$

$$\mathrm{Todd}(\tfrac{\partial}{\partial h_1})\,\mathrm{Todd}(\tfrac{\partial}{\partial h_2}) \int_{a-h_2}^{b+h_1} e^{zx}\,dx \Big|_{h_1=h_2=0} = \sum_{k=a}^{b} e^{kz}$$

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Theorem (MB–Gunnels–Materov) Let P be a simple lattice polytope. There is an (explicitly defined) differential operator $\mathrm{Todd}_{y,P}(\frac{\partial}{\partial h})$ such that

$$G_{\varphi,P}(t,y) \ = \ \mathrm{Todd}_{y,P}(\tfrac{\partial}{\partial h}) \left(\int_{P_{t,y}(h)} \varphi(x)\, dx \right) \Bigg|_{h=0} .$$

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- Euler–Maclaurin (ancient): $\varphi = 1$, $\dim(P) = 1$, contant term in y
- Khovanskii–Pukhlikov (1992): $\varphi = 1$, P smooth, contant term in y (closely related to the Hirzebruch-Riemann-Roch Theorem for smooth projective toric varieties)

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- Brion–Vergne (1997): P general lattice polytope, contant term in y

Extensions & Open Problems

- Rational polytopes & Ehrhart quasipolynomials
- Todd-operator formula for $G_{\varphi,P}(t,y)$ when P is not simple?
- Relation to Chapoton's q-Ehrhart polynomials?