Euler–Mahonian Statistics Via Polyhedral Geometry

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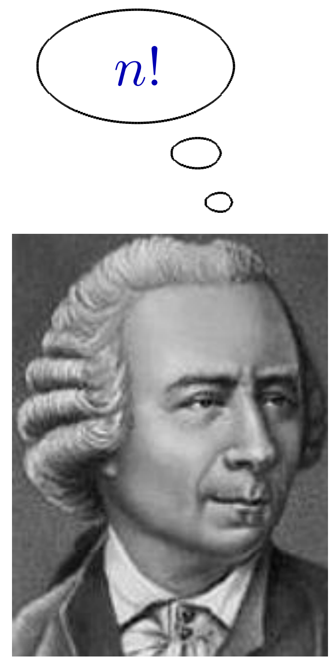

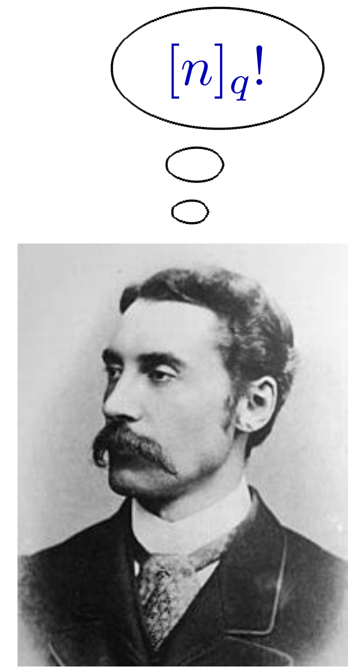

Permutation Statistics

$\pi \in S_n$ — permutation of $\{1, 2, \ldots, n\}$

Goal: study certain statistics of S_n (and other reflection groups), e.g.,

$$\begin{aligned} \mathrm{Des}(\pi) &:= \{j : \pi(j) > \pi(j+1)\} \\ \mathrm{des}(\pi) &:= \#\,\mathrm{Des}(\pi) \\ \mathrm{maj}(\pi) &:= \sum_{j \in \mathrm{Des}(\pi)} j \\ \mathrm{inv}(\pi) &:= \#\{(j,k) : j < k \text{ and } \pi(j) > \pi(k)\} \end{aligned}$$

Example $\quad S_3 = \{[123], [213], [132], [312], [132], [321]\}$

$$\sum_{\pi \in S_3} t^{\mathrm{des}(\pi)} = 1 + 4t + t^2$$

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More generally, for a Coxeter group W with generators $s_1, s_2, \dots, s_{n-1}$, the (right) descent set of $\sigma \in W$ is

$$\operatorname{Des}(\sigma) := \{j : l(\sigma s_j) < l(\sigma)\}$$

Permutation Statistics

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Sample Theorem 1 [Euler] $$\sum_{k \ge 0} (k+1)^n \, t^k = \frac{\sum_{\pi \in S_n} t^{\operatorname{des}(\pi)}}{(1-t)^{n+1}}$$

Sample Theorem 2 [MacMahon] $$\sum_{\pi \in S_n} t^{\operatorname{inv}(\pi)} = \sum_{\pi \in S_n} t^{\operatorname{maj}(\pi)}$$

Permutation Statistics

[Euler]

$$\sum_{k\geq 0}(k+1)^n\, t^k = \frac{\sum_{\pi\in S_n} t^{\operatorname{des}(\pi)}}{(1-t)^{n+1}}$$

[MacMahon]

$$\sum_{\pi\in S_n} t^{\operatorname{inv}(\pi)} = \sum_{\pi\in S_n} t^{\operatorname{maj}(\pi)}$$

Goal: new identities of this kind

Sample Theorem 3 [MacMahon]

$$\sum_{k\geq 0}[k+1]_q^n\, t^k = \frac{\sum_{\pi\in S_n} t^{\operatorname{des}(\pi)} q^{\operatorname{maj}(\pi)}}{\prod_{j=0}^{n}(1-tq^j)}$$

Sample Theorem 4 [Brenti, Steingrímsson]

$$\sum_{k\geq 0}(2k+1)^n\, t^k = \frac{\sum_{(\pi,\epsilon)\in B_n} t^{\operatorname{des}(\pi,\epsilon)}}{(1-t)^{n+1}}$$

(π,ϵ) — signed permutation with $\pi\in S_n$ and $\epsilon\in\{\pm 1\}$

Permutation Statistics

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[MacMahon]

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Sample Theorem 5 [Chow–Gessel]

$$\sum_{k \ge 0} \left([k+1]_q + s \, [k]_q \right)^n t^k = \frac{\sum_{(\pi,\epsilon) \in B_n} s^{\operatorname{neg}(\epsilon)} t^{\operatorname{des}(\pi,\epsilon)} q^{\operatorname{maj}(\pi,\epsilon)}}{\prod_{j=0}^{n} (1 - tq^j)}$$

(π, ϵ) — signed permutation with $\pi \in S_n$ and $\epsilon \in \{\pm 1\}$

Permutation Statistics

[Euler]

$$\sum_{k\geq 0}(k+1)^n\, t^k = \frac{\sum_{\pi\in S_n} t^{\mathrm{des}(\pi)}}{(1-t)^{n+1}}$$

[MacMahon]

$$\sum_{\pi\in S_n} t^{\mathrm{inv}(\pi)} = \sum_{\pi\in S_n} t^{\mathrm{maj}(\pi)}$$

Goal: new identities of this kind

- bijective proofs (integer partitions)
- Coxeter groups (invariant theory)
- symmetric & quasisymmetric functions
- polyhedral geometry

Enter Geometry

[Euler]

$$\sum_{k\geq 0}(k+1)^n\, t^k = \frac{\sum_{\pi\in S_n} t^{\operatorname{des}(\pi)}}{(1-t)^{n+1}}$$

[MacMahon]

$$\sum_{k\geq 0}[k+1]_q^n\, t^k = \frac{\sum_{\pi\in S_n} t^{\operatorname{des}(\pi)} q^{\operatorname{maj}(\pi)}}{\prod_{j=0}^{n}(1-tq^j)}$$

$\#\left(k\,[0,1]^n \cap \mathbb{Z}^n\right) = (k+1)^n$ is the Ehrhart polynomial of the unit n-cube

Use braid arrangement $\{x_j = x_k : \ 1 \leq j < k \leq n\}$ triangulation of $[0,1]^n$:

$$[0,1]^n = \bigcup_{\pi\in S_n}\left\{\boldsymbol{x}\in\mathbb{R}^n : \ 0 \leq x_{\pi(n)} \leq x_{\pi(n-1)} \leq \cdots \leq x_{\pi(1)} \leq 1 \ \right\}$$

Enter Geometry

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$$\sum_{k \ge 0} (k+1)^n \, t^k = \frac{\sum_{\pi \in S_n} t^{\operatorname{des}(\pi)}}{(1-t)^{n+1}}$$

[MacMahon]

$$\sum_{k \ge 0} [k+1]_q^n \, t^k = \frac{\sum_{\pi \in S_n} t^{\operatorname{des}(\pi)} q^{\operatorname{maj}(\pi)}}{\prod_{j=0}^{n} (1 - tq^j)}$$

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Enter Geometry

[Euler]

$$\sum_{k\geq 0}(k+1)^n\,t^k = \frac{\sum_{\pi\in S_n} t^{\operatorname{des}(\pi)}}{(1-t)^{n+1}}$$

[MacMahon]

$$\sum_{k\geq 0}[k+1]_q^n\,t^k = \frac{\sum_{\pi\in S_n} t^{\operatorname{des}(\pi)}q^{\operatorname{maj}(\pi)}}{\prod_{j=0}^n (1-tq^j)}$$

$\#\left(k\,[0,1]^n\cap\mathbb{Z}^n\right) = (k+1)^n$ is the Ehrhart polynomial of the unit n-cube

$$[0,1]^n = \bigsqcup_{\pi\in S_n}\left\{\boldsymbol{x}\in\mathbb{R}^n : \begin{array}{l} 0\leq x_{\pi(n)}\leq x_{\pi(n-1)}\leq\cdots\leq x_{\pi(1)}\leq 1, \\ x_{\pi(j+1)}<x_{\pi(j)} \text{ if } j\in\operatorname{Des}(\pi)\end{array}\right\}$$

Each simplex on the right is unimodular with Ehrhart series $\dfrac{t^{\#[\text{strict inequalities}]}}{(1-t)^{n+1}}$

$$\Longrightarrow\qquad \sum_{k\geq 0}(k+1)^n\,t^k = \sum_{\pi\in S_n}\frac{t^{\operatorname{des}(\pi)}}{(1-t)^{n+1}}$$

More Geometry

[Euler]

$$\sum_{k\geq 0}(k+1)^n\, t^k = \frac{\sum_{\pi\in S_n} t^{\mathrm{des}(\pi)}}{(1-t)^{n+1}}$$

[MacMahon]

$$\sum_{k\geq 0}[k+1]_q^n\, t^k = \frac{\sum_{\pi\in S_n} t^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)}}{\prod_{j=0}^n (1-tq^j)}$$

$$[0,1]^n = \bigsqcup_{\pi\in S_n} \left\{ \boldsymbol{x}\in\mathbb{R}^n : \begin{array}{l} 0\leq x_{\pi(n)}\leq x_{\pi(n-1)}\leq\cdots\leq x_{\pi(1)}\leq 1, \\ x_{\pi(j+1)} < x_{\pi(j)} \text{ if } j\in\mathrm{Des}(\pi) \end{array} \right\}$$

For $\mathcal{P}\subset\mathbb{R}^n$ consider $\sigma_{\mathrm{cone}(\mathcal{P})}(z_0,z_1,\dots,z_n) := \displaystyle\sum_{\boldsymbol{m}\in\mathrm{cone}(\mathcal{P})\cap\mathbb{Z}^{n+1}} z_0^{m_0} z_1^{m_1}\cdots z_n^{m_n}$

$$\begin{aligned} \sigma_{\mathrm{cone}([0,1]^n)}(z_0,z_1,\dots,z_n) &= \sum_{k\geq 0}\prod_{j=1}^n \left(1+z_j+z_j^2+\cdots+z_j^k\right) z_0^k \\ &= \sum_{k\geq 0}\prod_{j=1}^n [k+1]_{z_j}\, z_0^k \end{aligned}$$

More Geometry

[Euler]

$$\sum_{k\geq 0}(k+1)^n\, t^k = \frac{\sum_{\pi\in S_n} t^{\mathrm{des}(\pi)}}{(1-t)^{n+1}}$$

[MacMahon]

$$\sum_{k\geq 0}[k+1]_q^n\, t^k = \frac{\sum_{\pi\in S_n} t^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)}}{\prod_{j=0}^n (1-tq^j)}$$

$$[0,1]^n = \bigsqcup_{\pi\in S_n} \left\{ \boldsymbol{x}\in\mathbb{R}^n : \begin{array}{l} 0\leq x_{\pi(n)}\leq x_{\pi(n-1)}\leq\cdots\leq x_{\pi(1)}\leq 1, \\ x_{\pi(j+1)} < x_{\pi(j)} \text{ if } j\in\mathrm{Des}(\pi) \end{array} \right\}$$

Theorem

$$\sum_{k\geq 0}\prod_{j=1}^n [k+1]_{z_j}\, z_0^k = \sum_{\pi\in S_n} \frac{\prod_{j\in\mathrm{Des}(\pi)} z_0 z_{\pi(1)} z_{\pi(2)}\cdots z_{\pi(j)}}{\prod_{j=0}^n \left(1 - z_0\, z_{\pi(1)} z_{\pi(2)}\cdots z_{\pi(j)}\right)}$$

Remark Philosophy very close to that of P-partitions

MacMahon's theorem follows by setting $z_0 = t$ and $z_1 = z_2 = \cdots = z_n = q$

Type-B Permutation Statistics

(π, ϵ) — signed permutation with $\pi \in S_n$ and $\epsilon \in \{\pm 1\}$

Use the natural decent statistics

$$\begin{aligned} \mathrm{Des}(\pi) &:= \left\{ j : \epsilon_j \pi(j) > \epsilon_{j+1} \pi(j+1) \right\} \qquad [\epsilon_0 \pi(0) := 0] \\ \mathrm{des}(\pi) &:= \# \, \mathrm{Des}(\pi) \\ \mathrm{maj}(\pi) &:= \sum_{j \in \mathrm{Des}(\pi)} j \end{aligned}$$

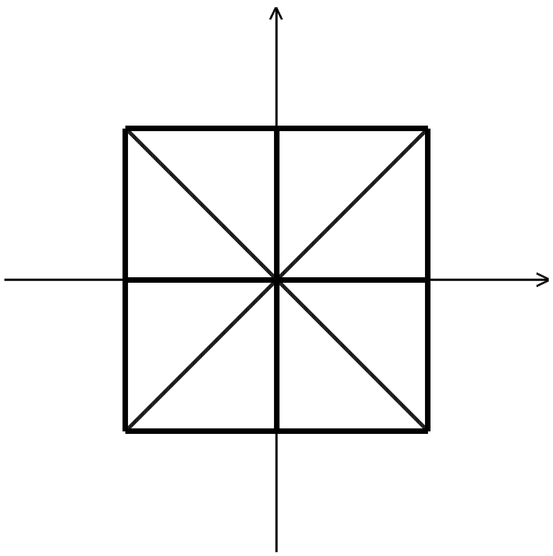
Sample Theorem 4 [Brenti, Steingrímsson]

$$\sum_{k \ge 0} (2k+1)^n \, t^k = \frac{\sum_{(\pi,\epsilon) \in B_n} t^{\mathrm{des}(\pi,\epsilon)}}{(1-t)^{n+1}}$$

Sample Theorem 5 [Chow–Gessel]

$$\sum_{k \ge 0} \left([k+1]_q + s \, [k]_q \right)^n t^k = \frac{\sum_{(\pi,\epsilon) \in B_n} s^{\mathrm{neg}(\epsilon)} t^{\mathrm{des}(\pi,\epsilon)} q^{\mathrm{maj}(\pi,\epsilon)}}{\prod_{j=0}^{n} (1 - t q^j)}$$

Type-B Geometry



[Brenti, Steingrímsson]

$$\sum_{k \geq 0} (2k+1)^n \, t^k = \frac{\sum_{(\pi,\epsilon) \in B_n} t^{\mathrm{des}(\pi,\epsilon)}}{(1-t)^{n+1}}$$

Use the type-B arrangement $\{x_j = \pm x_k, \ x_j = 0 : 1 \leq j < k \leq n\}$ to triangulate $[-1,1]^n$:

$$[-1,1]^n = \bigsqcup_{(\pi,\epsilon) \in B_n} \left\{ \boldsymbol{x} \in \mathbb{R}^n : \begin{array}{l} 0 \leq \epsilon_n x_{\pi(n)} \leq \epsilon_{n-1} x_{\pi(n-1)} \leq \cdots \leq \epsilon_1 x_{\pi(1)} \leq 1 \\ \epsilon_{j+1} x_{\pi(j+1)} < \epsilon_j x_{\pi(j)} \text{ if } j \in \mathrm{Des}(\pi,\epsilon) \end{array} \right\}$$

Each simplex on the right is unimodular with Ehrhart series $\dfrac{t^{\#[\text{strict inequalities}]}}{(1-t)^{n+1}}$

More Type-B Geometry

[Chow–Gessel]

$$\sum_{k\geq 0}\left([k+1]_q + s\,[k]_q\right)^n t^k = \frac{\sum_{(\pi,\epsilon)\in B_n} s^{\mathrm{neg}(\epsilon)} t^{\mathrm{des}(\pi,\epsilon)} q^{\mathrm{maj}(\pi,\epsilon)}}{\prod_{j=0}^{n}\left(1-tq^j\right)}$$

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Theorem

$$\sum_{k\geq 0}\prod_{j=1}^{n}\left(w_j [k+1]_{z_j} + w_{-j}\, z_{-j}^{-1} [k]_{z_{-j}^{-1}}\right) z_0^k =$$

$$\sum_{(\pi,\epsilon)\in B_n} \prod_{\epsilon_j=1} w_j \prod_{\epsilon_j=-1} z_{-j}^{-1} w_{-j} \frac{\prod_{j\in\mathrm{Des}(\pi,\epsilon)} z_0 z_{\epsilon_1\pi(1)}^{\epsilon_1} z_{\epsilon_2\pi(2)}^{\epsilon_2} \cdots z_{\epsilon_j\pi(j)}^{\epsilon_j}}{\prod_{j=0}^{n}\left(1 - z_0\, z_{\epsilon_1\pi(1)}^{\epsilon_1} z_{\epsilon_2\pi(2)}^{\epsilon_2} \cdots z_{\epsilon_j\pi(j)}^{\epsilon_j}\right)}$$

Chow–Gessel's theorem follows with $z_0 = t$, $z_1 = \cdots = z_n = z_{-1}^{-1} = \cdots = z_{-n}^{-1} = q$, $w_{-1} = \cdots = w_{-n} = s$, and $w_1 = \cdots = w_n = 1$

More Type-B Permutation Statistics

Using the total order $-1 < -2 < \cdots < -n < 1 < 2 < \cdots < n$, define $\operatorname{Des}(\pi,\epsilon)$, $\operatorname{des}(\pi,\epsilon)$, and $\operatorname{major}(\pi,\epsilon)$ as before, and define the negative descent multiset as

$$\begin{aligned}
\operatorname{NDes}(\pi,\epsilon) &:= \operatorname{Des}(\pi,\epsilon) \cup \{\pi(j) : \epsilon_j = -1\} \\
\operatorname{ndes}(\pi,\epsilon) &:= \#\operatorname{NDes}(\pi,\epsilon) \\
\operatorname{nmaj}(\pi,\epsilon) &:= \sum_{j \in \operatorname{NDes}(\pi,\epsilon)} j \\
\operatorname{fdes}(\pi,\epsilon) &:= 2 \cdot \operatorname{des}(\pi,\epsilon) + c_1 \qquad [\epsilon_1 = (-1)^{c_1}] \\
\operatorname{fmajor}(\pi,\epsilon) &:= 2 \cdot \operatorname{major}(\pi,\epsilon) + \operatorname{neg}(\pi,\epsilon)
\end{aligned}$$

Sample Theorems 6 & 7 [Adin–Brenti–Roichman]

$$\sum_{k \ge 0} [k+1]_q^n \, t^k = \frac{\sum_{(\pi,\epsilon)\in B_n} t^{[\operatorname{ndes}(\pi,\epsilon),\operatorname{fdes}(\pi,\epsilon)]} q^{[\operatorname{nmaj}(\pi,\epsilon),\operatorname{fmajor}(\pi,\epsilon)]}}{(1-t)\prod_{j=1}^n (1-t^2q^{2i})}$$

Even More Type-B Geometry

[Adin–Brenti–Roichman]

$$\sum_{k\geq 0} [k+1]_q^n \, t^k = \frac{\sum_{(\pi,\epsilon)\in B_n} t^{[\mathrm{ndes}(\pi,\epsilon),\mathrm{fdes}(\pi,\epsilon)]} q^{[\mathrm{nmaj}(\pi,\epsilon),\mathrm{fmajor}(\pi,\epsilon)]}}{(1-t)\prod_{j=1}^n (1-t^2q^{2i})}$$

Theorem Let $a_j^\epsilon := 1$ if $\epsilon_j \neq \epsilon_{j+1}$ and 0 otherwise. Then

$$\sum_{k\geq 0} \prod_{j=1}^n [k+1]_{z_j} z_0^k =$$

$$\sum_{(\pi,\epsilon)\in B_n} \frac{\prod_{\substack{j\in \mathrm{Des}(\pi)\\ a_j^\epsilon=0}} z_0^2 z_{\pi(1)}^2 z_{\pi(2)}^2 \cdots z_{\pi(j)}^2 \prod_{j:a_j^\epsilon=1} z_0 z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j)}}{(1-z_0)\prod_{j=1}^n \left(1 - z_0^2\, z_{\pi(1)}^2 z_{\pi(2)}^2 \cdots z_{\pi(j)}^2\right)}$$

Even More Type-B Geometry

$$\sum_{k\geq 0}\prod_{j=1}^{n}[k+1]_{z_j} z_0^k = \sum_{(\pi,\epsilon)\in B_n} \frac{\displaystyle\prod_{\substack{j\in \mathrm{Des}(\pi)\\ a_j^\epsilon=0}} z_0^2 z_{\pi(1)}^2 z_{\pi(2)}^2 \cdots z_{\pi(j)}^2 \prod_{j:a_j^\epsilon=1} z_0 z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(j)}}{(1-z_0)\displaystyle\prod_{j=1}^{n}\left(1 - z_0^2\, z_{\pi(1)}^2 z_{\pi(2)}^2 \cdots z_{\pi(j)}^2\right)}$$

Idea of Proof: Use the type-A triangulation

$$[0,1]^n = \bigsqcup_{\pi\in S_n} \left\{ \boldsymbol{x}\in\mathbb{R}^n : \begin{array}{l} 0\leq x_{\pi(n)}\leq x_{\pi(n-1)}\leq\cdots\leq x_{\pi(1)}\leq 1, \\ x_{\pi(j+1)} < x_{\pi(j)} \text{ if } j\in\mathrm{Des}(\pi) \end{array} \right\}$$

and the non-unimodular generators

$$\boldsymbol{e}_0, 2(\boldsymbol{e}_0+\boldsymbol{e}_{\pi(1)}), 2(\boldsymbol{e}_0+\boldsymbol{e}_{\pi(1)}+\boldsymbol{e}_{\pi(2)}), \ldots, 2(\boldsymbol{e}_0+\boldsymbol{e}_{\pi(1)}+\cdots+\boldsymbol{e}_{\pi(n)})$$

for the simplices on the right-hand side

Even More Type-B Geometry

[Adin–Brenti–Roichman]

$$\sum_{k \geq 0} [k+1]_q^n \, t^k = \frac{\sum_{(\pi,\epsilon) \in B_n} t^{[\mathrm{ndes}(\pi,\epsilon),\mathrm{fdes}(\pi,\epsilon)]} q^{[\mathrm{nmaj}(\pi,\epsilon),\mathrm{fmajor}(\pi,\epsilon)]}}{(1-t) \prod_{j=1}^n (1-t^2 q^{2i})}$$

Corollary Explicit bijection from B_n to itself (via integer lattice points) that preserves the pairs of statistics $(\mathrm{ndes}, \mathrm{nmaj})$ and $(\mathrm{fdes}, \mathrm{fmajor})$

(Another bijection was previously given by Lai–Petersen.)

Type-D Permutation Statistics

$D_n := \{(\pi, \epsilon) \in B_n : \epsilon_1 \epsilon_2 \cdots \epsilon_n = 1\}$

Define the natural decent statistics as in type B except that now we use the convention $\epsilon_0 \pi(0) := -\epsilon_2 \pi(2)$

Sample Theorem 8 [Brenti]

$$\sum_{k \ge 0} \left((2k+1)^n - 2^{n-1} \left(\mathcal{B}_n(k+1) - \mathcal{B}_n(0) \right) \right) t^k = \frac{\sum_{(\pi,\epsilon) \in D_n} t^{\operatorname{des}(\pi,\epsilon)}}{(1-t)^{n+1}}$$

where $\mathcal{B}_n(x)$ is the nth Bernoulli polynomial. Equivalently,

$$\sum_{k \ge 0} (2k+1)^n \, t^k = \frac{\sum_{(\pi,\epsilon) \in D_n} t^{\operatorname{des}(\pi,\epsilon)} + t^{1+\operatorname{des}_2(\pi,\epsilon)}}{(1-t)^{n+1}}$$

where $\operatorname{des}_2(\pi, \epsilon) := \# \left(\operatorname{Des}(\pi, \epsilon) \cap [2, n] \right)$.

What about a q-analogue in the spirit of Chow–Gessel?

Type-D Geometry

[Brenti]
$$\sum_{k \ge 0} (2k+1)^n \, t^k = \frac{\sum_{(\pi,\epsilon) \in D_n} t^{\operatorname{des}(\pi,\epsilon)} + t^{1+\operatorname{des}_2(\pi,\epsilon)}}{(1-t)^{n+1}}$$

Theorem
$$\sum_{k \ge 0} \left([k+1]_q + s\,[k]_q \right)^n t^k =$$
$$\frac{\sum_{(\pi,\epsilon) \in D_n} s^{N_2(\epsilon)} t^{\operatorname{des}_2(\pi,\epsilon)} q^{\operatorname{maj}_2(\pi,\epsilon)} \left((tq)^{[0 \text{ or } 1 \in \operatorname{Des}(\pi,\epsilon)]} + st(tq)^{[0 \text{ and } 1 \in \operatorname{Des}(\pi,\epsilon)]} \right)}{\prod_{j=0}^{n} (1 - tq^j)}$$

Idea of Proof Combine two of the type-B-triangulation simplices at a time

Brenti's theorem follows upon setting $s = q = 1$ and noticing that
$$t^{\operatorname{des}_2(\pi,\epsilon)} \left(t^{[0 \text{ or } 1 \in \operatorname{Des}(\pi,\epsilon)]} + t \cdot t^{[0 \text{ and } 1 \in \operatorname{Des}(\pi,\epsilon)]} \right) = t^{\operatorname{des}(\pi,\epsilon)} + t^{1+\operatorname{des}_2(\pi,\epsilon)}$$

What else can be geometrized?

Sample Theorem 9 [Biagioli]

$$\sum_{k\ge 0}[k+1]_q^n\, t^k = \frac{\sum_{(\pi,\epsilon)\in D_n} t^{\mathrm{ndes}(\pi,\epsilon)} q^{\mathrm{nmaj}(\pi,\epsilon)}}{(1-t)(1-tq^n)\prod_{j=1}^{n-1}(1-t^2q^{2i})}$$

Sample Theorem 10 & 11 [Bagno, Bagno–Biagioli]

$$\sum_{k\ge 0}[k+1]_q^n\, t^k = \frac{\sum_{(\pi,\epsilon)\in \mathbb{Z}_r\wr S_n} t^{[\mathrm{ndes}(\pi,\epsilon),\mathrm{fdes}(\pi,\epsilon)]} q^{[\mathrm{nmaj}(\pi,\epsilon),\mathrm{fmajor}(\pi,\epsilon)]}}{(1-t)\prod_{j=1}^{n}(1-t^rq^{ri})}$$

Sample Theorem 12 [similar to Chow–Mansour]

$$\sum_{k\ge 0}[rk+1]_q^n\, t^k = \frac{\sum_{(\pi,\epsilon)\in \mathbb{Z}_r\wr S_n} t^{\mathrm{des}(\pi,\epsilon)} q^{\mathrm{fmajor}(\pi,\epsilon)}}{\prod_{j=0}^{n}(1-tq^{rj})}$$

The Message

- Unifying proofs of Euler–Mahonian statistics results through discrete polyhedral geometry

- Multivariate generalizations [Corollary: Hilbert-series interpretations]

- Bijective proofs of the equidistribution of various pairs of statistics