Partitions with Fixed Differences Between Largest and Smallest Parts

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Integer Partitions

A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{Z}^k$ of an integer $n > 0$ satisfies

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k \qquad \text{and} \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$$

Example

$$\begin{aligned} 5 &= 1+1+1+1+1 \\ &= 2+1+1+1 \\ &= 2+2+1 \\ &= 3+1+1 \\ &= 3+2 \\ &= 4+1 \\ &= 5 \end{aligned}$$

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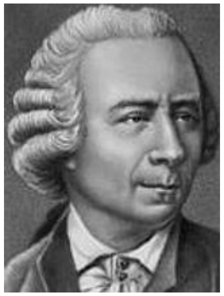
- Number Theory
- Combinatorics
- Symmetric functions
- Representation Theory
- Physics





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Main Goal Understand $p(n,t) := \#$ partitions of n with $\lambda_1 - \lambda_k = t$

Integer Partitions With Fixed Difference 2...

The On-Line Encyclopedia of Integer Sequences®
founded in 1964 by N. J. A. Sloane

Many excellent designs for a new banner were submitted. We will use the best of them in rotation.

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A008805 Triangular numbers repeated.

1, 1, 3, 3, 6, 6, 10, 10, 15, 15, 21, 21, 28, 28, 36, 36, 45, 45, 55, 55, 66, 66, 78, 78, 91, 91, 105, 105, 120, 120, 136, 136, 153, 153, 171, 171, 190, 190, 210, 210, 231, 231, 253, 253, 276, 276, 300, 300, 325 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

COMMENTS Number of choices for nonnegative integers x,y,z such that x and y are even and x+y+z = n.

a(n) = number of partitions of n+4 such that the differences between greatest and smallest parts are 2: a(n-4) = A097364(n,2) for n>3. - Reinhard Zumkeller, Aug 09 2004

Integer Partitions With Fixed Difference 3...

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Search Hints

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A128508 Number of partitions p of n such that max(p)-min(p)=3.

0, 0, 0, 0, 0, 1, 1, 3, 3, 7, 7, 12, 14, 20, 22, 32, 34, 45, 51, 63, 69, 87, 93, 112, 124, 144, 156, 184, 196, 225, 245, 275, 295, 335, 355, 396, 426, 468, 498, 552, 582, 637, 679, 735, 777, 847, 889, 960, 1016, 1088, 1144, 1232, 1288, 1377, 1449, 1539, 1611, 1719 (list; graph; refs; listen; history; text; internal format)

OFFSET	0,8
COMMENTS	See A008805 and A049820 for the numbers of partitions p of n such that max(p)-min(p)=1 or 2, respectively.
LINKS	Alois P. Heinz, Table of n, a(n) for n = 0..1000
FORMULA	Conjecture. a(1)=0 and, for n>1, a(n+1)=a(n)+d(n), where d(n) is defined as follows: d=0,0,0,1,0 for n=1,...,5 and, for n>5, d(n)=d(n-2)+1 if n=6k or n=6k+4, d(n)=d(n-2) if n=6k+1 or n=6k+3, d(n)=d(n-2)+2Floor[n/6] if n=6k+2 and d(n)=d(n-5) if n=6k+5.

... to 10

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Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A218573 Number of partitions p of n such that max(p)-min(p) = 10.

1, 1, 3, 3, 7, 8, 14, 18, 28, 35, 53, 67, 93, 119, 161, 201, 267, 332, 428, 531, 674, 824, 1034, 1258, 1552, 1877, 2294, 2749, 3332, 3970, 4762, 5645, 6723, 7916, 9367, 10974, 12894, 15036, 17571, 20381, 23696, 27370, 31652, 36416, 41926, 48029, 55071, 62860 (list; graph; refs; listen; history; text; internal format)

OFFSET	12,3
LINKS	Alois P. Heinz, Table of n, a(n) for n = 12..1000
FORMULA	G.f.: Sum_{k>0} x^(2*k+10)/Product_{j=0..10} (1-x^(k+j)). a(n) = A097364(n,10) = A116685(n,10) = A194621(n,10) - A194621(n,9) = A218512(n) - A218511(n).
CROSSREFS	Sequence in context: A218570 A218571 A218572 * A117989 A241642 A086543 Adjacent sequences: A218570 A218571 A218572 * A218574 A218575 A218576

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Main Goal Understand $p(n,t) := \#$ partitions of n with $\lambda_1 - \lambda_k = t$

Equivalently, understand $P_t(q) := \sum_{n \ge 1} p(n,t)\, q^n$

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Integer Partitions With Fixed Difference 2...

Quasipolynomials

A quasipolynomial is a function $\mathbb{Z} \to \mathbb{R}$ of the form

$$q(k) \;=\; c_d(k)\, k^d + c_{d-1}(k)\, k^{d-1} + \cdots + c_0(k)$$

where $c_0(k), \dots, c_d(k)$ are periodic functions. Equivalently,

$$\sum_{k \ge 0} q(k)\, z^k \;=\; \frac{h(z)}{(1 - z^p)^{d+1}}$$

for some (minimal) $p \in \mathbb{Z}_{>0}$, where $\deg(h(z)) < (d+1)p$

Example $\;P_2(q) \;=\; \dfrac{q^4}{(1-q)^3(1+q)^2} \;=\; \dfrac{q^4 + q^5}{(1-q^2)^3}$

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$$p(n,2) \ = \ \begin{cases} \frac{n^2}{8} - \frac{n}{4} & \text{if } n \text{ is even} \\ \frac{n^2}{8} - \frac{n}{2} + \frac{3}{8} & \text{if } n \text{ is odd} \end{cases} \ = \ \binom{\lfloor \frac{n}{2} \rfloor}{2}$$

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Example $P_3(q) = \dfrac{q^5 + q^6 + q^7 - q^8}{(1-q^2)^2(1-q^3)^2}$

$$p(n,3) = \frac{1}{108} \times \begin{cases} n^3 - 18n & \text{if } n \equiv 0 \bmod 6 \\ n^3 - 3n + 2 & \text{if } n \equiv 1 \bmod 6 \\ n^3 - 30n + 52 & \text{if } n \equiv 2 \bmod 6 \\ n^3 + 9n - 54 & \text{if } n \equiv 3 \bmod 6 \\ n^3 - 30n + 56 & \text{if } n \equiv 4 \bmod 6 \\ n^3 - 3n - 2 & \text{if } n \equiv 5 \bmod 6 \end{cases}$$

Main Results

$p(n,t) :=$ # partitions of n with $\lambda_1 - \lambda_k = t$ $\qquad P_t(q) := \sum_{n \ge 1} p(n,t)\, q^n$

Theorem (Andrews–MB–Robbins 2015) For $t > 1$

$$P_t(q) = \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})} - \frac{q^{t-1}}{(1-q^t)^2(1-q^{t-1})^2(1-q^{t-2})\cdots(1-q^2)} + \frac{q^t}{(1-q^t)(1-q^{t-1})^2(1-q^{t-2})\cdots(1-q)}$$

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Corollary If $t > 1$ then $p(n,t) = \frac{n^t}{t\,(t!)^2} + O(n^{t-1})$ as $n \to \infty$.

Main Results

$p_{\le}(n,t) :=$ # partitions of n with $\lambda_1 - \lambda_k \le t$

$$P_{\le t}(q) := \sum_{n \ge 1} p_{\le}(n,t)\, q^n$$

Corollary (Breuer–Kronholm 2016) For $t > 0$

$$P_{\le t}(q) \ = \ \left(\frac{1}{(1-q)(1-q^2)\cdots(1-q^t)} - 1 \right) \frac{1}{1-q^t}$$

Partitions With Specified Distances

$p(n, t_1, t_2, \dots, t_k) := \#$ partitions of n such that, if σ is the smallest part then $\sigma + t_1 + t_2 + \cdots + t_k$ is the largest part and each of $\sigma + t_1, \ \sigma + t_1 + t_2, \ \dots, \ \sigma + t_1 + t_2 + \cdots + t_{k-1}$ appear as parts.

$$P_{t_1,\dots,t_k}(q) \ := \ \sum_{n \ge 1} p(n, t_1, t_2, \dots, t_k) \, q^n$$

Theorem (Andrews–MB–Robbins 2015)

$$P_{t_1,\dots,t_k}(q) \ = \ \frac{(-1)^k q^{T - \binom{k+1}{2}} \left(\sum_{j=0}^{k} \begin{bmatrix} t \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2}} - (q)_t \right)}{\begin{bmatrix} t-1 \\ k \end{bmatrix} (1 - q^t)(q)_t}$$

where $t := t_1 + \cdots + t_k > k$ and $T := kt_1 + (k-1)t_2 + \cdots + 2t_{k-1} + t_k$.

Here $(A)_m := (1 - A)(1 - A\,q) \cdots (1 - A\,q^{m-1})$ and $\begin{bmatrix} n \\ k \end{bmatrix} := \frac{(q)_n}{(q)_k (q)_{n-k}}$

Proof Idea

$$
\begin{aligned}
P_2(q) \quad &= \quad \sum_{m \geq 1} \frac{q^m}{1-q^m} \, \frac{1}{1-q^{m+1}} \, \frac{q^{m+2}}{1-q^{m+2}} \\
&= \quad q^2 \sum_{m \geq 1} \frac{q^{2m} (q)_{m-1}}{(q)_{m+2}} \;=\; \frac{q^4}{(q)_3} \sum_{m \geq 1} \frac{q^{2m} (q)_m (q)_m}{(q)_m (q^4)_m} \\
&= \quad \frac{q^4 (q^3)_\infty (q^3)_\infty}{(q)_3 (q^4)_\infty (q^2)_\infty} \sum_{j \geq 0} \frac{q^{3j} (q)_j}{(q)_j (q^3)_j} \;=\; \frac{q^4 (1-q^3)}{(q)_3 (1-q^2)}
\end{aligned}
$$

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\begin{aligned}
P_2(q) &= \sum_{m\ge 1} \frac{q^m}{1-q^m}\,\frac{1}{1-q^{m+1}}\,\frac{q^{m+2}}{1-q^{m+2}} \\
&= q^2 \sum_{m\ge 1} \frac{q^{2m}(q)_{m-1}}{(q)_{m+2}} \;=\; \frac{q^4}{(q)_3} \sum_{m\ge 1} \frac{q^{2m}(q)_m(q)_m}{(q)_m(q^4)_m} \\
&= \frac{q^4(q^3)_\infty(q^3)_\infty}{(q)_3(q^4)_\infty(q^2)_\infty} \sum_{j\ge 0} \frac{q^{3j}(q)_j}{(q)_j(q^3)_j} \;=\; \frac{q^4(1-q^3)}{(q)_3(1-q^2)}
\end{aligned}
$$

Heine's Transform $\displaystyle \sum_{m\ge 0} \frac{(a)_m(b)_m\, z^m}{(q)_m(c)_m} = \frac{(\frac{c}{b})_\infty(bz)_\infty}{(c)_\infty(z)_\infty} \sum_{j\ge 0} \frac{(\frac{abz}{c})_j(b)_j(\frac{c}{b})^j}{(q)_j(bz)_j}$

Extensions

- Breuer–Kronholm (2016): polyhedral model
- Chapman (2016): elementary proof
- Chern (2017): 3-variable generalization
- Chern (2017), Chern–Yee (2018): overpartitions
- Berkovich–Uncu (2019): partition inequalities
- Lin (2020): refinement by number of parts

Quasipolynomials in Nature

Very Basic Problem Given $\Phi \in \mathbb{Z}^{r \times m}$ (of rank r), enumerate all solutions $\mathbf{x} \in \mathbb{Z}_{\geq 0}^{m}$ to the system of eqations $\Phi\,\mathbf{x} = \mathbf{0}$.

These solutions form a semigroup S. If $\mathbf{x} \in S$ satisfies

$$n\,\mathbf{x} = \mathbf{y} + \mathbf{y}' \qquad \Longrightarrow \qquad \mathbf{y} = j\,\mathbf{x}, \;\; \mathbf{y} = (n-j)\,\mathbf{x}$$

for any $n \in \mathbb{Z}_{>0}$ and $\mathbf{y}, \mathbf{y}' \in S$ then $\mathbf{x}$ is completely fundamental. We collect the completely fundamental elements of S in the set $\mathrm{CF}(S)$.

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Theorem (Stanley 1973) The generating function $\displaystyle\sum_{\mathbf{x} \in S} \mathbf{z}^{\mathbf{x}} = \sum_{\mathbf{x} \in S} z_1^{x_1} \cdots z_m^{x_m}$ can be written as a rational function with denominator $\displaystyle\prod_{\mathbf{x} \in \mathrm{CF}(S)} (1 - \mathbf{z}^{\mathbf{x}})$.

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My Favorite Interpretation S are the integer lattice points in the rational cone $\left\{\mathbf{x} \in \mathbb{R}_{\geq 0}^{d} : \, \mathbf{A}\,\mathbf{x} = \mathbf{0}\right\}$

Partitions Done Geometrically

$P_{\le t}(q) := \sum_{n \ge 1} \#(\text{partitions of } n \text{ with } \lambda_1 - \lambda_k \le t)\, q^n$

Corollary (Breuer–Kronholm 2016) For $t > 0$

$$\begin{aligned} P_{\le t}(q) &= \sum_{m \ge 1} \frac{q^m}{(1-q^m)(1-q^{m+1})\cdots(1-q^{m+t})} \\ &= \left(\frac{1}{(1-q)(1-q^2)\cdots(1-q^t)} - 1 \right) \frac{1}{1-q^t} \end{aligned}$$

Natural Question Is there a (geometric) reason why this infinite sum of rational functions simplifies to a single rational function?

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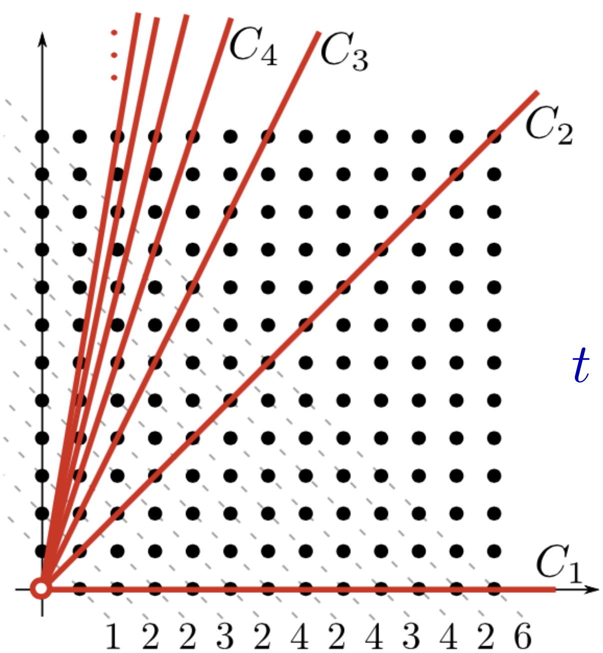
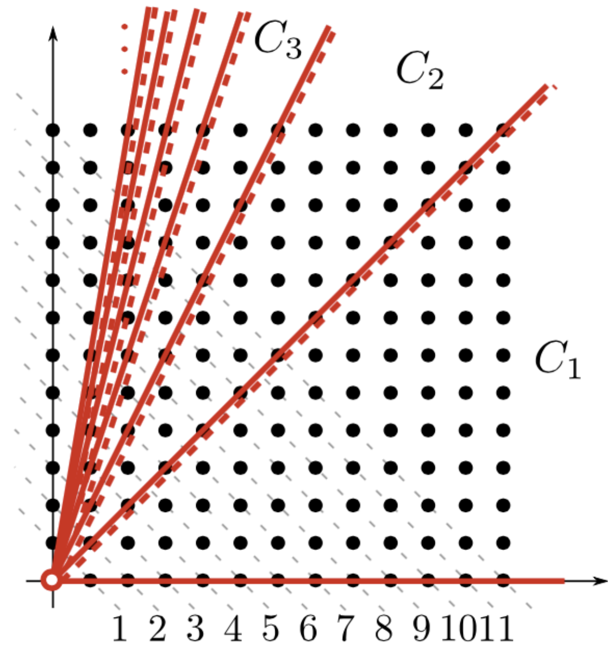


$t = 0$

$t = 1$



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- Follows from a polyhedral model: partitions are precisely the integer points in a $t+1$-dimensional (half-open, simplicial) cone.

- Leads to a natural bijective proof and...

Theorem (Breuer–Kronholm 2016) $p_{\leq}(n,t)$ equals the number of pairs (λ, k) where $k \geq 0$ is divisible by t and λ is a non-empty partition of $n-k$ with largest part at most t.