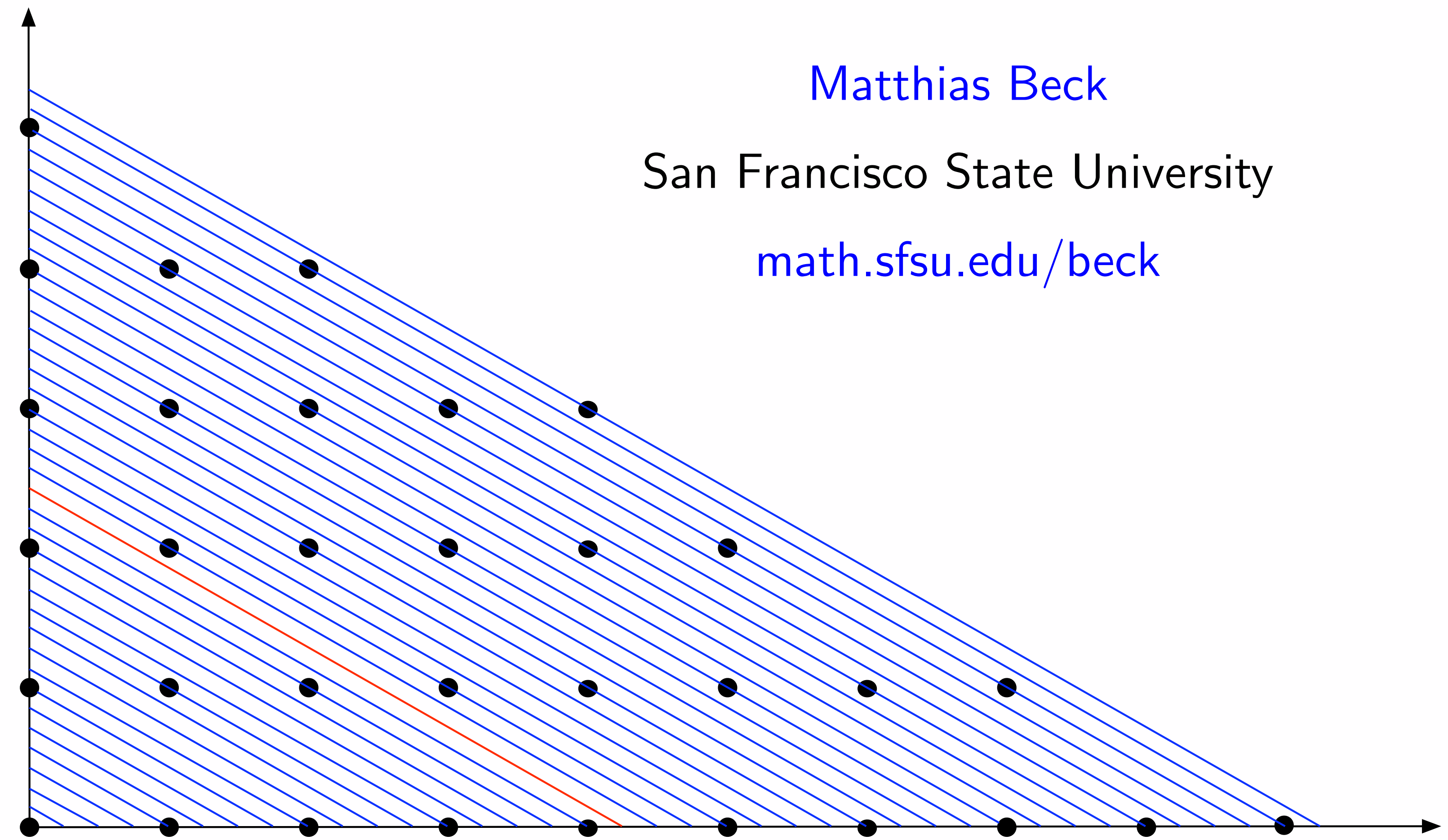
The "Coin Exchange Problem" of Frobenius



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The problem

Given coins of denominations $a_1, a_2, \ldots, a_d$ (with no common factor), what is the largest amount that cannot be changed?

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- $d = 3$ solved algorithmically (Herzog 1970, Greenberg 1980, Davison 1994) and in not-quite-explicit form (Denham 2003, Ramirez-Alfonsin 2005)

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- $d = 2$ solved (probably by Sylvester in 1880's)
- $d = 3$ solved algorithmically (Herzog 1970, Greenberg 1980, Davison 1994) and in not-quite-explicit form (Denham 2003, Ramirez-Alfonsin 2005)
- $d \geq 4$ computationally feasible (Kannan 1992, Barvinok-Woods 2003), otherwise: completely open

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One of the coefficients tm and tn is negative.

Claim: If t is sufficiently large then we can express it as a nonnegative integral linear combination of a and b.

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$$t = (m + 81b)a + (n - 81a)b \ .$$

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Once we have one such representation of t we can find many more:

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There is a unique representation

$$t = ma + nb$$

for which $0 \leq m \leq b - 1$.

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for which $0 \le m \le b - 1$. So if t is large enough, e.g., $\ge ab$, then we can find a nonnegative integral linear combination of a and b.

A well-defined homework

Prove that the Frobenius problem is well defined for $d > 2$.

A closer look for two coins

Given two positive integers a and b with no common factor, we say the integer t is representable if we can find nonnegative integers m and n such that

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for which $0 \leq m \leq b - 1$.

If (and only if) we can find such a representation for which also $n \geq 0$ then t is representable. Hence the largest integer t that is not representable is

$$t = ab - a - b \ ,$$

a formula most likely known already to Sylvester in the 1880's.

A homework with several representations

Given two positive integers a and b with no common factor, we say the integer t is k-representable if there are exactly k solutions $(m,n) \in \mathbb{Z}_{\geq 0}^2$ to

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- $g_k = (k+1)ab - a - b$

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- g_k is well defined.
- $g_k = (k+1)ab - a - b$
- Given $k \geq 2$, the smallest k-representable integer is $ab(k-1)$.

Further consequences

Let $N(t) = \#\left\{(m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ ma + nb = t\right\}$

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Starting from this representation, we can obtain more...

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$$t = (m + 7b)a + (n - 7a)b$$

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has one more representation, i.e., $N(t+ab) = N(t) + 1$.

A geometric interlude

$N(t) = \#\left\{(m,n) \in \mathbb{Z}^2 : \; m,n \geq 0, \; ma + nb = t\right\}$ counts integer points in $\mathbb{R}^2_{\geq 0}$ on the line $ax + by = t$.

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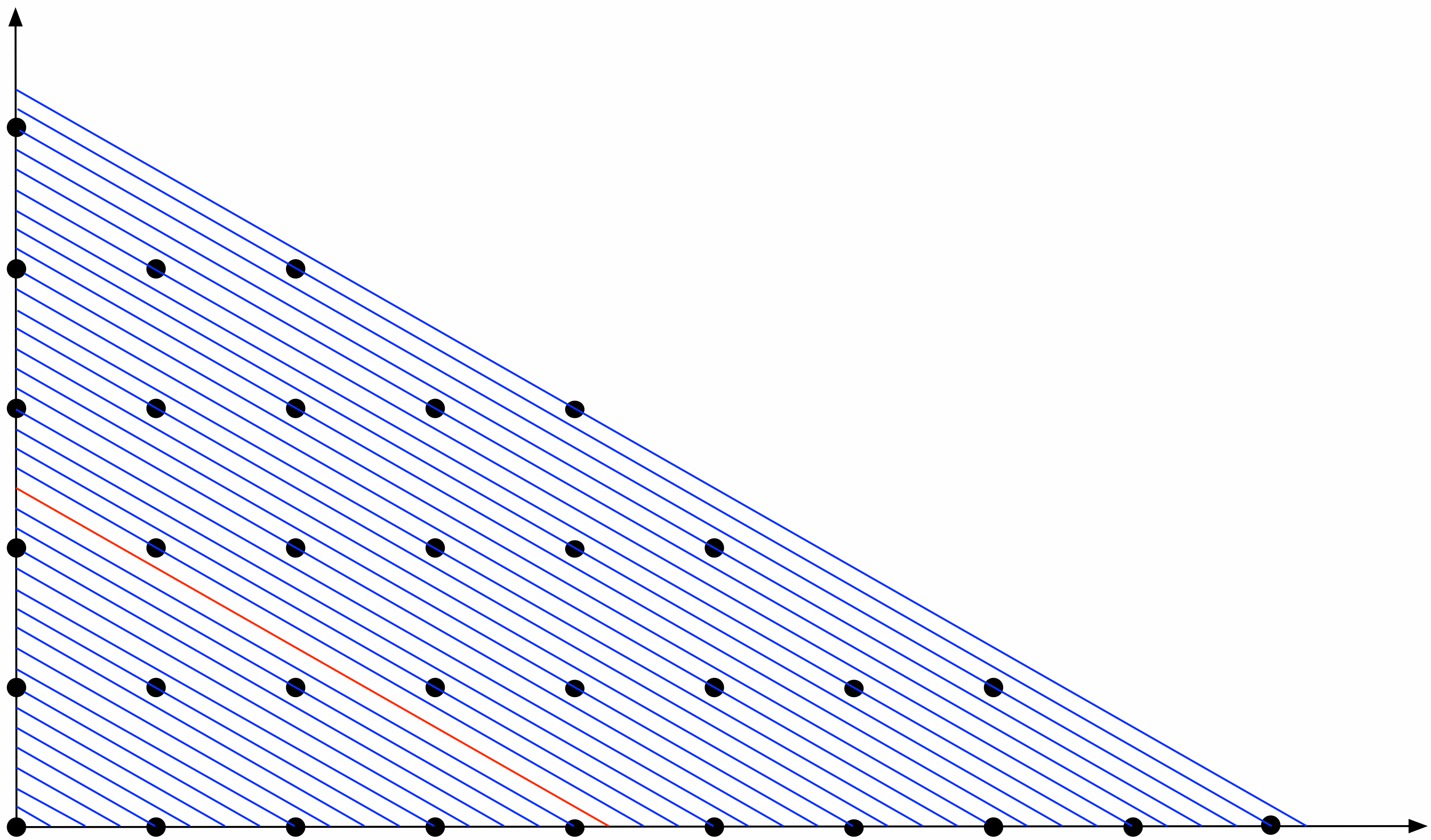

Shameless plug

M. Beck & S. Robins

Computing the continuous discretely
Integer-point enumeration in polyhedra

To appear in Springer Undergraduate Texts in Mathematics

Preprint available at math.sfsu.edu/beck

MSRI Summer Graduate Program at Banff (August 6–20)

Generating functions

Given a sequence $(a_k)_{k=0}^{\infty}$ we encode it into the generating function

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$$\begin{aligned} \sum_{k \geq 0} f_{k+2} \, x^k &= \sum_{k \geq 0} f_{k+1} \, x^k + \sum_{k \geq 0} f_k \, x^k \\ \frac{1}{x^2} \sum_{k \geq 2} f_k \, x^k &= \frac{1}{x} \sum_{k \geq 1} f_k \, x^k + \sum_{k \geq 0} f_k \, x^k \end{aligned}$$

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\frac{1}{x^2} \sum_{k \ge 2} f_k \, x^k &= \frac{1}{x} \sum_{k \ge 1} f_k \, x^k + \sum_{k \ge 0} f_k \, x^k \\
\frac{1}{x^2} \left(F(x) - x \right) &= \frac{1}{x} F(x) + F(x)
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$f_0 = 0, \; f_1 = 1,$ and $f_{k+2} = f_{k+1} + f_k$ for $k \geq 0$.

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\frac{1}{x^2} \left(F(x) - x \right) &= \frac{1}{x} F(x) + F(x) \\
F(x) &= \frac{x}{1 - x - x^2} \, .
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A Fibonacci homework

Expand $F(x) = \sum_{k \geq 0} f_k \, x^k = \frac{x}{1 - x - x^2}$ into partial fractions...

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$$f_k = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right) .$$

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$$\sum_{k \geq 0} a_k \, x^k = \sum_{j \geq 0} x^{7j} = \frac{1}{1 - x^7}$$

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Consider the product of the geometric series

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A typical term looks like x^{ma+nb} for some $m, n \geq 0$

Frobenius generatingfunctionology

Given positive integers a and b, recall that

$$N(t) = \#\left\{(m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ ma + nb = t\right\}.$$

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$$\frac{1}{(1-x^a)(1-x^b)} = \sum_{t \geq 0} N(t)\, x^t$$

is the generating function associated to the counting function $N(t)$.

More Frobenius generatingfunctionology

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and

$$\sum_{t \text{ not representable}} x^t = \frac{1}{1-x} - \frac{1-x^{ab}}{(1-x^a)\,(1-x^b)} \, .$$

A higher-dimensional homework

We say that t is representable by the positive integers $a_1, a_2, \ldots, a_d$ if there is a solution $(m_1, m_2, \ldots, m_d)$ in nonnegative integers to

$$m_1 a_1 + m_2 a_2 + \cdots + m_d a_d = t$$

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Prove

$$\sum_{t \text{ representable}} x^t = \frac{p(x)}{(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_d})}$$

for some polynomial p.

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Another homework with several representations

Recall: Given two positive integers a and b with no common factor, we say the integer t is k-representable if there are exactly k solutions $(m,n) \in \mathbb{Z}^2_{\geq 0}$ to

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Prove:

- There are exactly $ab - 1$ integers that are uniquely representable.
- Given $k \geq 2$, there are exactly ab k-representable integers.

Beyond d=2

Given integers $a_1, a_2, \ldots, a_d$ with no common factor, let

$$F(x) = \sum_{t \text{ representable}} x^t \;=\; \frac{p(x)}{(1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_d})}\,.$$

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- (Barvinok-Woods 2003) For fixed d, the rational generating function $F(x)$ can be written as a "short" sum of rational functions.

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Open problems

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A few references

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