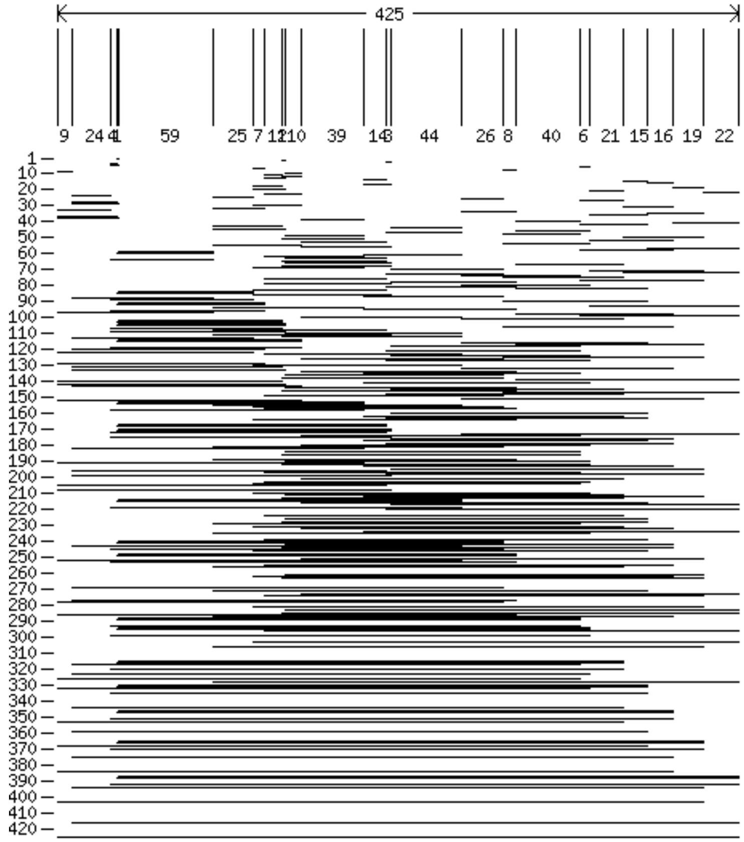
Golomb Rulers

Matthias Beck
San Francisco State University

Tristram Bogart
Universidad de los Andes

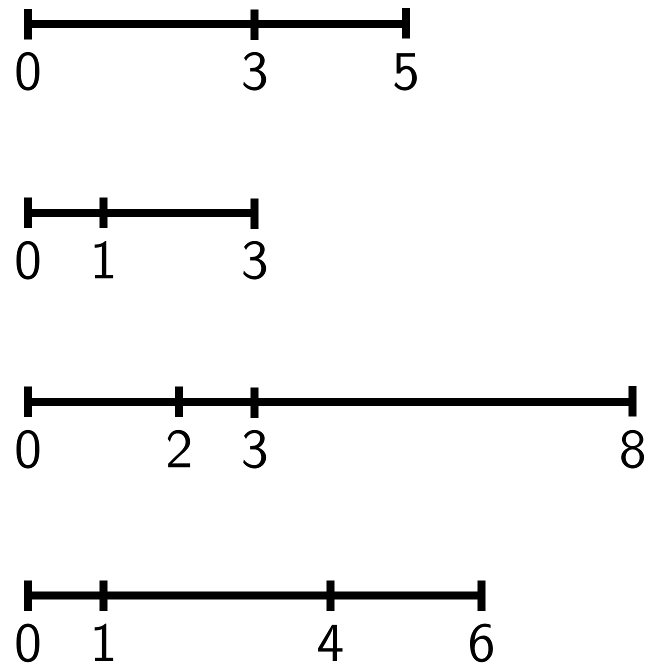
Tu Pham
UC Riverside

arXiv:1110.6154



OGR-24 Optimal Ruler - http://distributed.net/ogr

All Sorts of Golomb Rulers



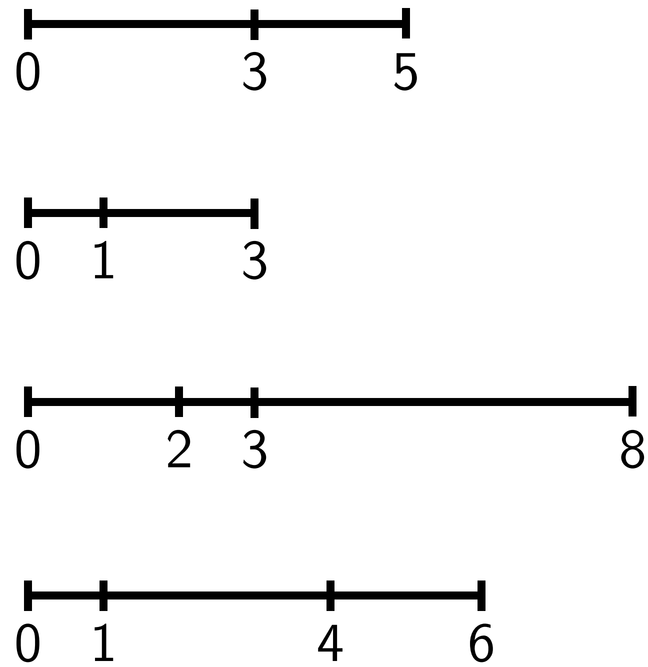
Golomb ruler: sequence of distinct integers with distinct pairwise differences

Every Golomb ruler comes with a length t and some $m+1$ markings

Optimal Golomb rulers have minimal length for a given number of markings

Perfect Golomb rulers can measure every integer from 0 to t

All Sorts of Golomb Rulers



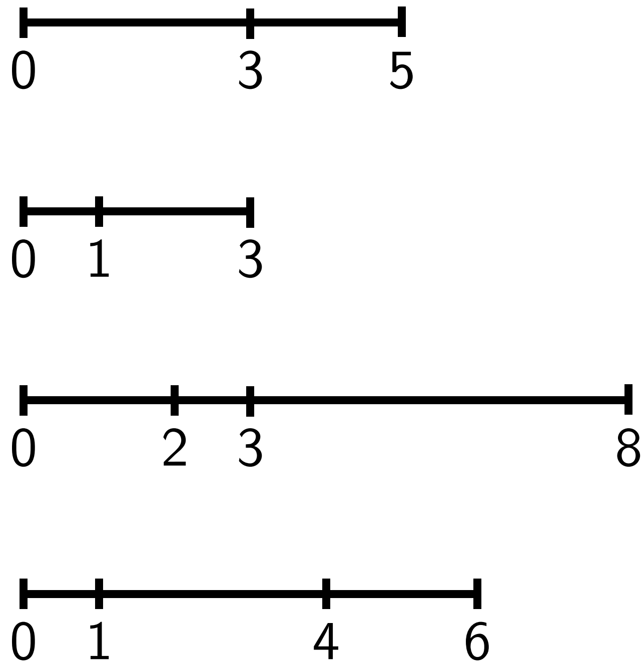
Every Golomb ruler comes with a length t and $m+1$ markings

Optimal Golomb rulers have minimal length for a given number of markings

Perfect Golomb rulers can measure every integer from 0 to t

Fun Exercise: There are no perfect Golomb rulers of lenth $t > 6$

All Sorts of Golomb Rulers



Every Golomb ruler comes with a length t and $m+1$ markings

Research problem: find optimal Golomb rulers with > 26 markings (see `http://www.distributed.net/OGR` for computational results).

Our goal: count all Golomb rulers for given t and m

Motivations & Applications

- Distortion problems in consecutive radio bands $\longrightarrow$ place radio signals so that all distances are distinct (Babcock 1950's)
- Error-correcting codes
- Additive number theory (Sidon sets)
- Dissonant music pieces (see Scott Rickard's TED talk)

Enumeration of Golomb Rulers

Goal Study/compute the number $g_m(t)$ of Golomb rulers of length t with $m+1$ markings

Example $\qquad g_2(t) = \#\left\{ x \in \mathbb{Z} : 0 < x < t, \; x \neq t - x \right\}$

Enumeration of Golomb Rulers

Goal Study/compute the number $g_m(t)$ of Golomb rulers of length t with $m+1$ markings

Example $$g_2(t) = \#\left\{x \in \mathbb{Z} : 0 < x < t,\ t \neq 2x\right\}$$

Enumeration of Golomb Rulers

Goal Study/compute the number $g_m(t)$ of Golomb rulers of length t with $m+1$ markings

Example 1 $$g_2(t) = \#\left\{ x \in \mathbb{Z} : 0 < x < t, \ t \neq 2x \right\}$$

$$= \begin{cases} t-1 & \text{if } t \text{ is odd} \\ t-2 & \text{if } t \text{ is even} \end{cases}$$

... a quasipolynomial in t

Example 2

$$g_3(t) = \begin{cases} \frac{1}{2}t^2 - 4t + 10 & \text{if } t \equiv 0, \\ \frac{1}{2}t^2 - 3t + \frac{5}{2} & \text{if } t \equiv 1, 5, 7, 11, \\ \frac{1}{2}t^2 - 4t + 6 & \text{if } t \equiv 2, 10, \\ \frac{1}{2}t^2 - 3t + \frac{9}{2} & \text{if } t \equiv 3, 9, \\ \frac{1}{2}t^2 - 4t + 8 & \text{if } t \equiv 4, 6, 8 \end{cases} \pmod{12}$$

Enumeration of Golomb Rulers

Goal Study/compute the number $g_m(t)$ of Golomb rulers of length t with $m+1$ markings

Example 1 $\quad g_2(t) = \#\left\{x \in \mathbb{Z} : 0 < x < t,\ t \neq 2x\right\}$

$$= \begin{cases} t-1 & \text{if } t \text{ is odd} \\ t-2 & \text{if } t \text{ is even} \end{cases}$$

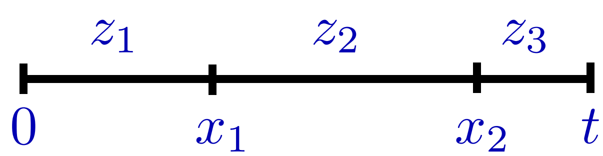
. . . a quasipolynomial in t

Theorem 1 The Golomb counting function $g_m(t)$ is a quasipolynomial in t of degree $m-1$ with leading coefficient $\frac{1}{(m-1)!}$

Let's start counting. . .

$$g_3(t) \quad := \quad \#\left\{ \boldsymbol{x} \in \mathbb{Z}^4 : \begin{array}{l} 0 = x_0 < x_1 < x_2 < x_3 = t \\ \text{all } x_j - x_k \text{ distinct} \end{array} \right\}$$

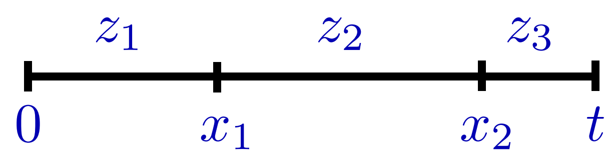
Let's start counting. . .



$$g_3(t) := \#\left\{ \boldsymbol{x} \in \mathbb{Z}^4 : \begin{array}{l} 0 = x_0 < x_1 < x_2 < x_3 = t \\ \text{all } x_j - x_k \text{ distinct} \end{array} \right\}$$

$$= \#\left\{ \boldsymbol{z} \in \mathbb{Z}^3_{>0} : \begin{array}{l} z_1 + z_2 + z_3 = t \\ \sum_{j \in U} z_j \neq \sum_{j \in V} z_j \text{ for all dpcs } U, V \subset [3] \end{array} \right\}$$

Let's start counting. . .



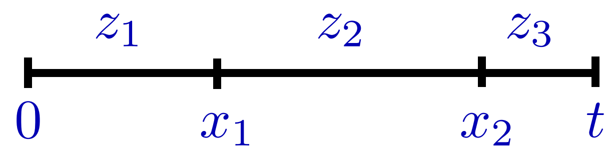
$$g_3(t) \quad := \quad \# \left\{ \boldsymbol{x} \in \mathbb{Z}^4 : \begin{array}{l} 0 = x_0 < x_1 < x_2 < x_3 = t \\ \text{all } x_j - x_k \text{ distinct} \end{array} \right\}$$

$$= \quad \# \left\{ \boldsymbol{z} \in \mathbb{Z}^3_{>0} : \begin{array}{l} z_1 + z_2 + z_3 = t \\ \sum_{j \in U} z_j \neq \sum_{j \in V} z_j \text{ for all dpcs } U, V \subset [3] \end{array} \right\}$$

where dpcs is shorthand for "disjoint proper consecutive subset," and $[m] := \{1, 2, \ldots, m\}$.

$$\begin{aligned} x_1 \neq x_2 &\iff z_2 > 0 \\ x_2 \neq t - x_1 &\iff z_1 \neq z_3 \\ x_2 \neq t - x_2 &\iff z_1 + z_2 \neq z_3 \end{aligned}$$

Let's start counting. . .



$$\begin{aligned} g_3(t) \quad &:= \quad \# \left\{ \boldsymbol{x} \in \mathbb{Z}^4 : \begin{array}{l} 0 = x_0 < x_1 < x_2 < x_3 = t \\ \text{all } x_j - x_k \text{ distinct} \end{array} \right\} \\ &= \quad \# \left\{ \boldsymbol{z} \in \mathbb{Z}^3_{>0} : \begin{array}{l} z_1 + z_2 + z_3 = t \\ \sum_{j \in U} z_j \neq \sum_{j \in V} z_j \text{ for all dpcs } U, V \subset [3] \end{array} \right\} \end{aligned}$$

where dpcs is shorthand for "disjoint proper consecutive subset," and $[m] := \{1, 2, \ldots, m\}$. More generally,

$$\begin{aligned} g_m(t) \quad &:= \quad \# \left\{ \boldsymbol{x} \in \mathbb{Z}^{m+1} : \begin{array}{l} 0 = x_0 < x_1 < \cdots < x_{m-1} < x_m = t \\ \text{all } x_j - x_k \text{ distinct} \end{array} \right\} \\ &= \quad \# \left\{ \boldsymbol{z} \in \mathbb{Z}^m_{>0} : \begin{array}{l} z_1 + z_2 + \cdots + z_m = t \\ \sum_{j \in U} z_j \neq \sum_{j \in V} z_j \text{ for all dpcs } U, V \subset [m] \end{array} \right\} \end{aligned}$$

Enter Geometry

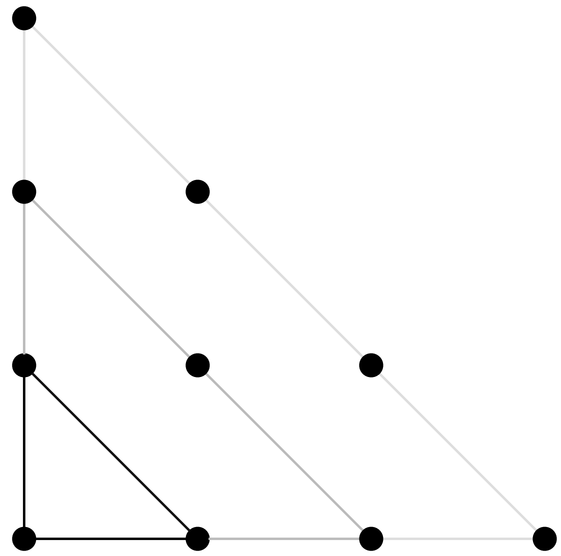
Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$

Enter Geometry

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$



Example:

$$\Delta = \operatorname{conv}\{(0,0),(1,0),(0,1)\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 : \; x,y \geq 0, \; x+y \leq 1 \right\}$$

$$L_\Delta(t) = \dots$$

Enter Geometry

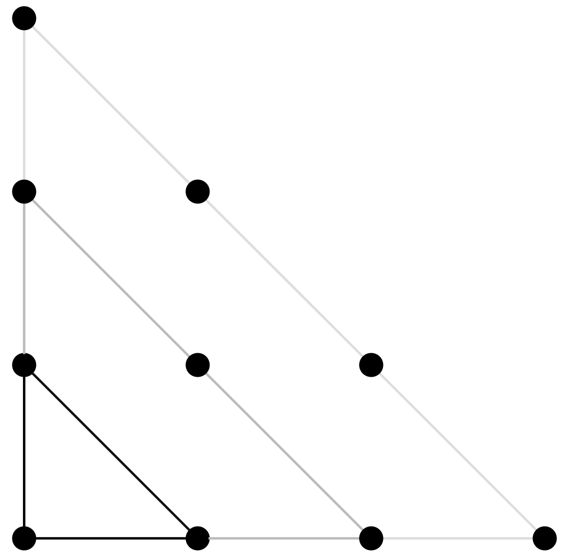
Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$

Example:

$$\Delta = \operatorname{conv}\{(0,0),(1,0),(0,1)\}$$
$$= \left\{ (x,y) \in \mathbb{R}^2 : \, x,y \geq 0, \, x+y \leq 1 \right\}$$

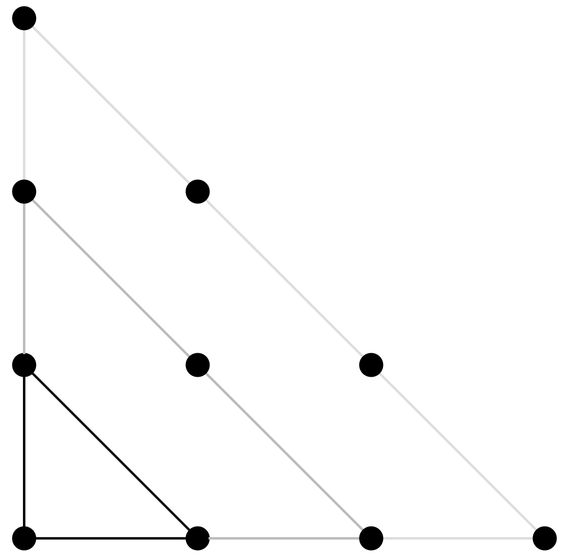
$$L_\Delta(t) = \binom{t+2}{2} = \tfrac{1}{2}(t+1)(t+2)$$



Enter Geometry

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$



Example:

$$\Delta = \operatorname{conv} \{(0,0),(1,0),(0,1)\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 : \, x,y \geq 0, \, x+y \leq 1 \right\}$$

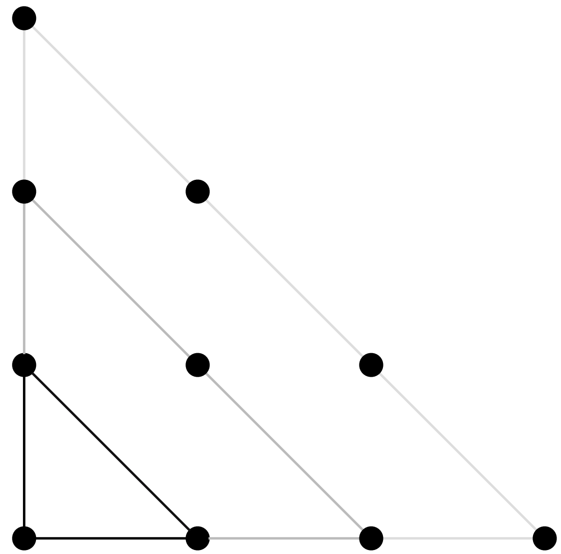
$$L_\Delta(t) = \binom{t+2}{2} = \tfrac{1}{2}(t+1)(t+2)$$

$$L_\Delta(-t) = \binom{t-1}{2}$$

Enter Geometry

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$



Example:

$$\Delta = \operatorname{conv} \{(0,0),(1,0),(0,1)\}$$
$$= \left\{ (x,y) \in \mathbb{R}^2 : \, x,y \geq 0, \, x+y \leq 1 \right\}$$

$$L_\Delta(t) = \binom{t+2}{2} = \tfrac{1}{2}(t+1)(t+2)$$

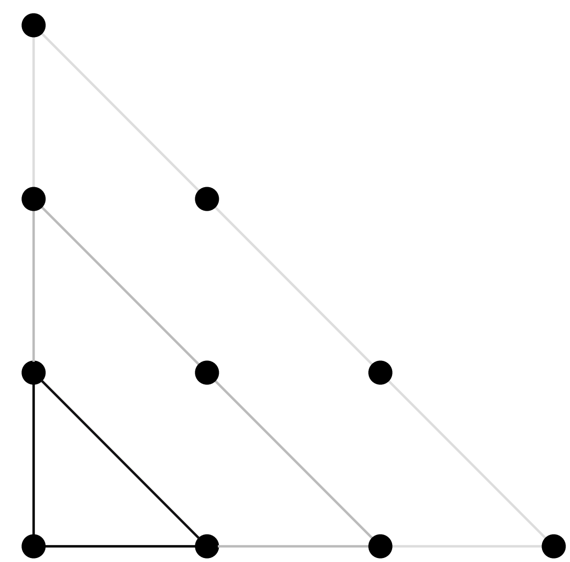
$$L_\Delta(-t) = \binom{t-1}{2} = L_{\Delta^\circ}(t)$$

For example, the evaluations $L_\Delta(-1) = L_\Delta(-2) = 0$ point to the fact that neither Δ nor 2Δ contain any interior lattice points.

Enter Geometry

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$



Example:

$$\Delta = \operatorname{conv}\left\{(0,0),(1,0),(0,1)\right\}$$
$$= \left\{(x,y) \in \mathbb{R}^2 : \, x, y \geq 0, \, x + y \leq 1\right\}$$

$$L_\Delta(t) = \binom{t+2}{2} = \frac{1}{2}(t+1)(t+2)$$

$$L_\Delta(-t) = \binom{t-1}{2} = L_{\Delta^\circ}(t)$$

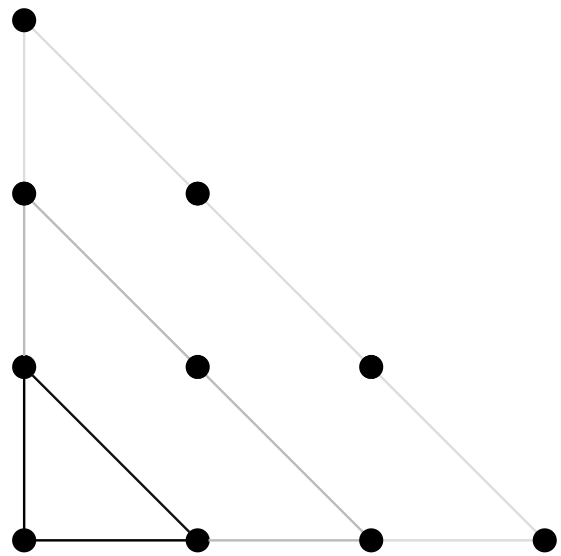
Theorem (Ehrhart 1962) $L_{\mathcal{P}}(t)$ is a polynomial in t.

Theorem (Macdonald 1971) $(-1)^{\dim \mathcal{P}} L_{\mathcal{P}}(-t) = L_{\mathcal{P}^\circ}(t)$

Enter Geometry

Rational polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Q}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$



Example:

$$\Delta = \operatorname{conv}\{(0,0),(1,0),(0,1)\}$$
$$= \left\{ (x,y) \in \mathbb{R}^2 : \, x,y \geq 0, \, x+y \leq 1 \right\}$$

$$L_\Delta(t) = \binom{t+2}{2} = \frac{1}{2}(t+1)(t+2)$$

$$L_\Delta(-t) = \binom{t-1}{2} = L_{\Delta^\circ}(t)$$

Theorem (Ehrhart 1962) $L_{\mathcal{P}}(t)$ is a quasipolynomial in t.

Theorem (Macdonald 1971) $(-1)^{\dim \mathcal{P}} L_{\mathcal{P}}(-t) = L_{\mathcal{P}^\circ}(t)$

Enter Geometry

Rational polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Q}^d$

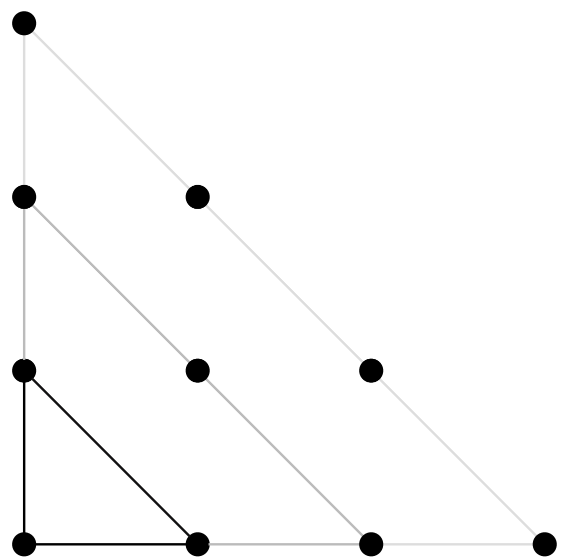
For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$

Example:

$$\Delta = \operatorname{conv}\{(0,0),(1,0),(0,1)\}$$
$$= \left\{ (x,y) \in \mathbb{R}^2 : \, x, y \geq 0, \, x + y \leq 1 \right\}$$

$$L_\Delta(t) = \binom{t+2}{2} = \tfrac{1}{2}(t+1)(t+2)$$

$$L_\Delta(-t) = \binom{t-1}{2} = L_{\Delta^\circ}(t)$$

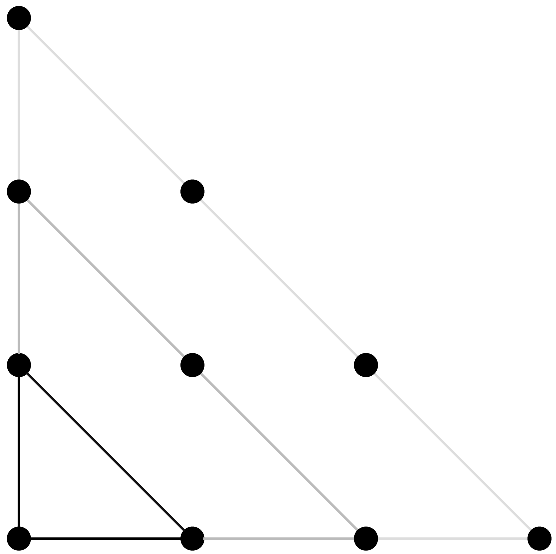


For 2-dimensional lattice polygons, Ehrhart–Macdonald's theorem follows from Pick's theorem.

Enter Geometry

Rational polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Q}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$



Example:

$$\Delta = \operatorname{conv}\{(0,0),(1,0),(0,1)\}$$
$$= \left\{ (x,y) \in \mathbb{R}^2 : \, x,y \geq 0, \, x+y \leq 1 \right\}$$

$$L_\Delta(t) = \# \left\{ (x,y) \in \mathbb{Z}^2_{\geq 0} : \, x+y \leq t \right\}$$

Enter Geometry

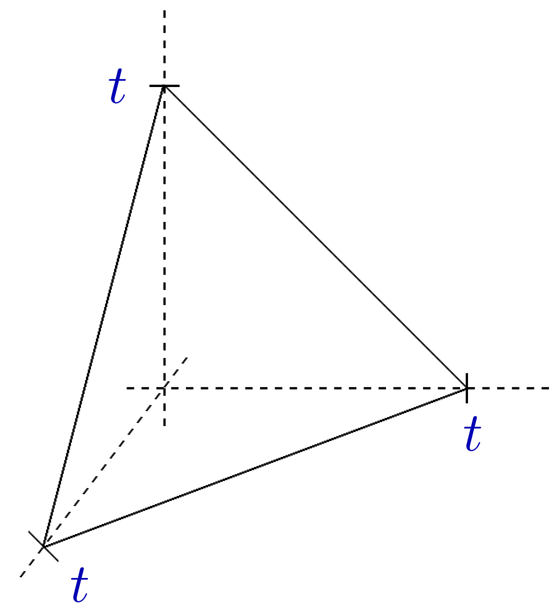
Rational polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Q}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right)$

Example:

$$\Delta = \operatorname{conv}\{(0,0),(1,0),(0,1)\}$$
$$= \left\{(x,y) \in \mathbb{R}^2 : \, x, y \geq 0, \, x + y \leq 1\right\}$$

$$L_\Delta(t) = \#\left\{(x,y) \in \mathbb{Z}^2_{\geq 0} : \, x + y \leq t\right\}$$
$$= \#\left\{(x,y,z) \in \mathbb{Z}^3_{\geq 0} : \, x + y + z = t\right\}$$

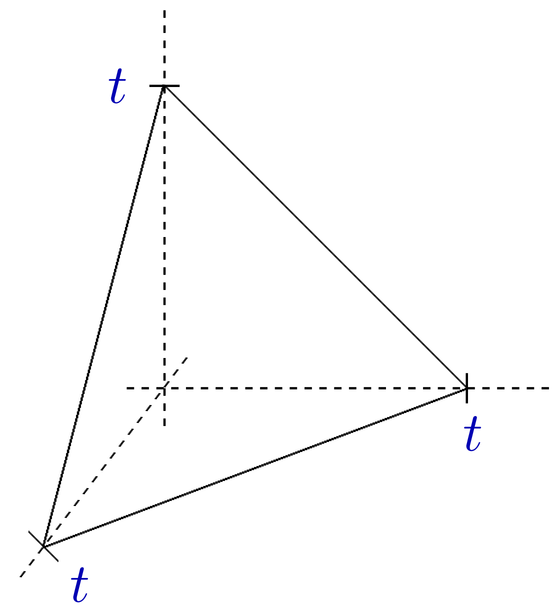


Enter Geometry

Rational polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Q}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$

Example:



$$\Delta = \operatorname{conv}\{(0,0),(1,0),(0,1)\}$$
$$= \left\{ (x,y) \in \mathbb{R}^2 : \, x,y \ge 0, \, x+y \le 1 \right\}$$

$$L_\Delta(t) = \# \left\{ (x,y) \in \mathbb{Z}^2_{\ge 0} : \, x+y \le t \right\}$$
$$= \# \left\{ (x,y,z) \in \mathbb{Z}^3_{\ge 0} : \, x+y+z = t \right\}$$

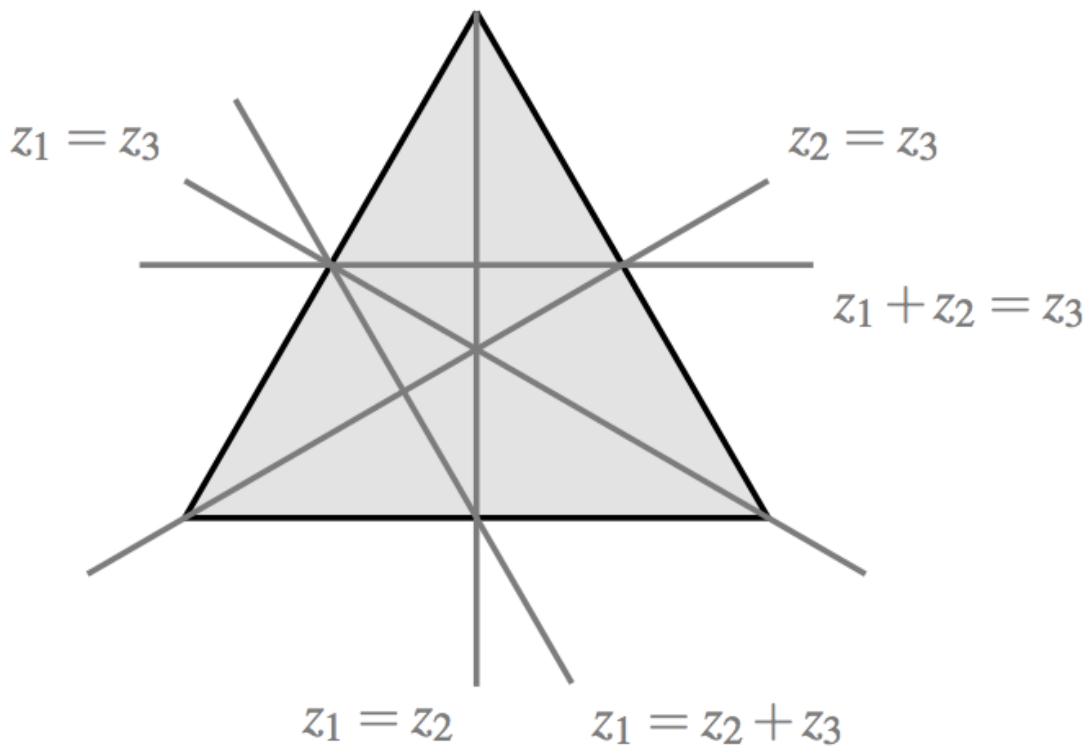
Hmmm . . .

$$g_3(t) = \# \left\{ \boldsymbol{z} \in \mathbb{Z}^3_{>0} : \begin{array}{l} z_1 + z_2 + z_3 = t \\ \sum_{j \in U} z_j \ne \sum_{j \in V} z_j \text{ for all dpcs } U, V \subset [3] \end{array} \right\}$$

The Geometry Behind Golomb Rulers

$$g_3(t) \quad := \quad \# \left\{ \boldsymbol{x} \in \mathbb{Z}^4 : \begin{array}{l} 0 = x_0 < x_1 < x_2 < x_3 = t \\ \text{all } x_j - x_k \text{ distinct} \end{array} \right\}$$

$$= \quad \# \left\{ \boldsymbol{z} \in \mathbb{Z}^3_{>0} : \begin{array}{l} z_1 + z_2 + z_3 = t \\ \sum_{j \in U} z_j \neq \sum_{j \in V} z_j \text{ for all dpcs } U, V \subset [3] \end{array} \right\}$$



The Geometry Behind Golomb Rulers

$$g_m(t) \quad := \quad \# \left\{ \boldsymbol{x} \in \mathbb{Z}^{m+1} : \begin{array}{l} 0 = x_0 < x_1 < \cdots < x_{m-1} < x_m = t \\ \text{all } x_j - x_k \text{ distinct} \end{array} \right\}$$

$$= \quad \# \left\{ \boldsymbol{z} \in \mathbb{Z}^m_{>0} : \begin{array}{l} z_1 + z_2 + \cdots + z_m = t \\ \sum_{j \in U} z_j \neq \sum_{j \in V} z_j \text{ for all dpcs } U, V \subset [m] \end{array} \right\}$$

. . . counts integer points in t-dilates of the m-dimensional simplex

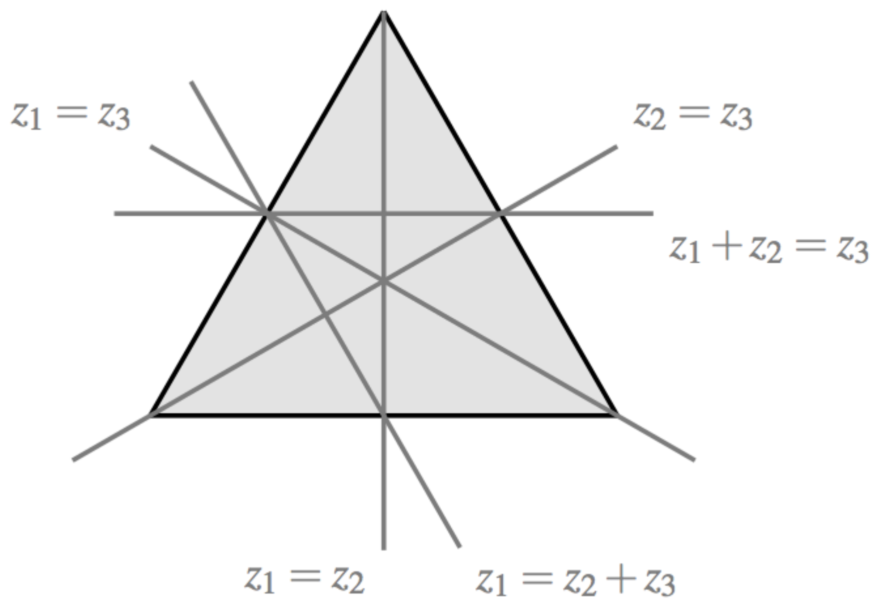
$$\Delta_m^\circ := \{ \boldsymbol{z} \in \mathbb{R}^m_{>0} : z_1 + z_2 + \cdots + z_m = 1 \}$$

that are off the hyperplanes

$$\sum_{j \in U} z_j = \sum_{j \in V} z_j$$

for all dpcs $U, V \subset [m]$

(This gives Theorem 1.)



Real Golomb Rulers

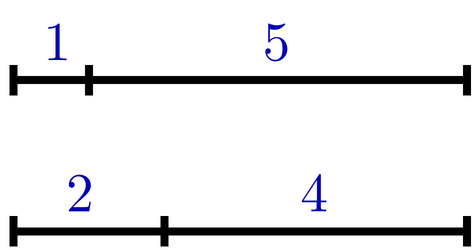
Real Golomb ruler — $\boldsymbol{z} \in \mathbb{R}_{\geq 0}^m$ satisfying $z_1 + z_2 + \cdots + z_m = t$ and

$$\sum_{j \in U} z_j \neq \sum_{j \in V} z_j \text{ for all dpcs } U, V \subset [m]$$

$\boldsymbol{z}, \mathbf{w} \in \mathbb{R}_{\geq 0}^m$ are combinatorially equivalent if for any dpcs $U, V \subset [m]$

$$\sum_{j \in U} z_j < \sum_{j \in V} z_j \qquad \Longleftrightarrow \qquad \sum_{j \in U} w_j < \sum_{j \in V} w_j$$

i.e., if their possible measurements satisfy the same order relations.



More Geometry Behind Golomb Rulers

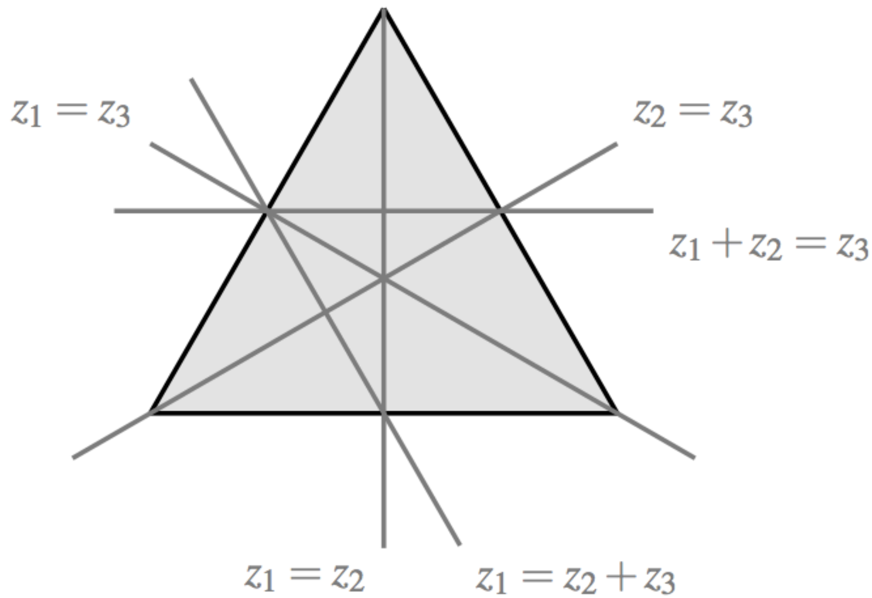
Recall the Golomb inside-out polytope formed by the simplex

$$\Delta_m^\circ := \{\boldsymbol{z} \in \mathbb{R}^m_{>0} : \, z_1 + z_2 + \cdots + z_m = 1\}$$

and the hyperplanes

$$\sum_{j \in U} z_j = \sum_{j \in V} z_j$$

for all dpcs $U, V \subset [m]$. Its regions correspond to combinatorially different Golomb rulers



More Geometry Behind Golomb Rulers

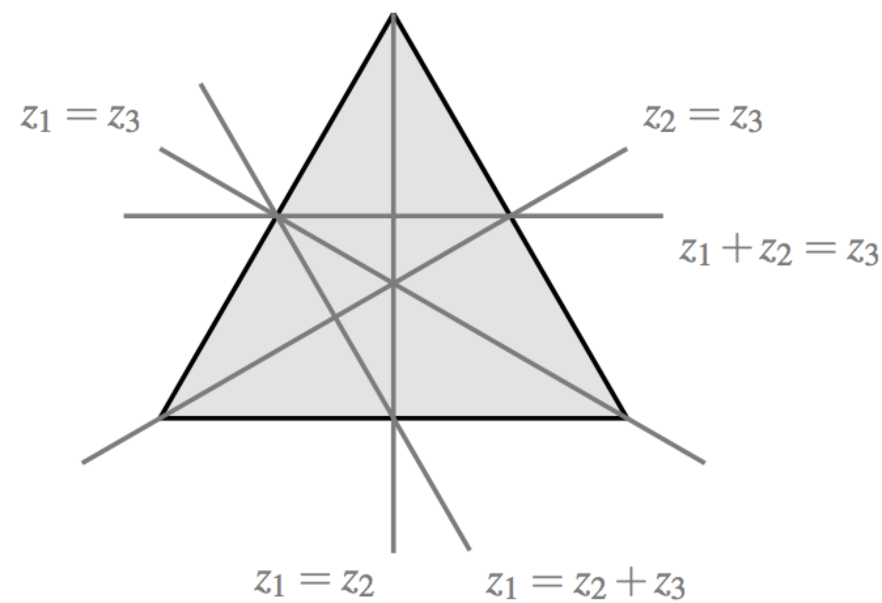
Recall the Golomb inside-out polytope formed by the simplex

$$\Delta_m^\circ := \{ \boldsymbol{z} \in \mathbb{R}^m_{>0} : \, z_1 + z_2 + \cdots + z_m = 1 \}$$

and the hyperplanes

$$\sum_{j \in U} z_j = \sum_{j \in V} z_j$$

for all dpcs $U, V \subset [m]$. Its regions correspond to combinatorially different Golomb rulers



Theorem 2 $(-1)^{m-1} g_m(0)$ equals the number of combinatorially different Golomb rulers with $m+1$ markings.

(This follows from $L_{\mathcal{P}}(0) = 1$ for any Ehrhart quasipolynomial. . .)

More Geometry Behind Golomb Rulers

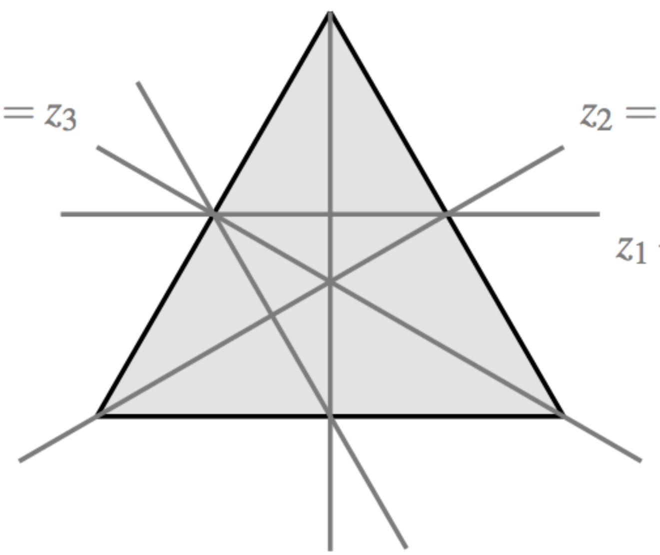
Recall the Golomb inside-out polytope formed by the simplex

$$\Delta_m^\circ := \{ \boldsymbol{z} \in \mathbb{R}^m_{>0} : \, z_1 + z_2 + \cdots + z_m = 1 \}$$

and the hyperplanes

$$\sum_{j \in U} z_j = \sum_{j \in V} z_j$$

for all dpcs $U, V \subset [m]$. Its regions correspond to combinatorially different Golomb rulers



Theorem 2 $(-1)^{m-1} g_m(0)$ equals the number of combinatorially different Golomb rulers with $m+1$ markings.

Have you seen this sequence? $1, 2, 10, 114, 2608, 107498, \ldots$

Golomb Ruler Reciprocity

$\boldsymbol{z}, \mathbf{w} \in \mathbb{R}^m_{\geq 0}$ are combinatorially equivalent if for any dpcs $U, V \subset [m]$

$$\sum_{j \in U} z_j < \sum_{j \in V} z_j \qquad \Longleftrightarrow \qquad \sum_{j \in U} w_j < \sum_{j \in V} w_j$$

Golomb multiplicty of $\boldsymbol{z} \in \mathbb{Z}^m_{\geq 0}$ — number of combinatorially different real Golomb rulers in an ϵ-neighborhood of $\boldsymbol{z}$

Theorem 3 $(-1)^{m-1} g_m(-t)$ equals the number of rulers in $\mathbb{Z}^m_{\geq 0}$ of length t each counted with its Golomb multiplicity.

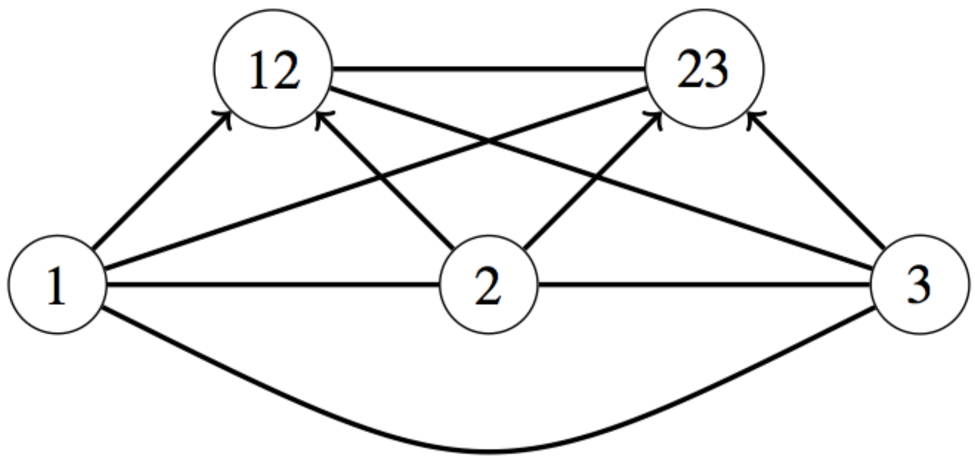
(This follows from Ehrhart–Macdonald reciprocity. . .)

Golomb Graphs

G_m — mixed graph whose vertices are all proper consecutive subsets of $[m]$

Underlying graph is complete and $U \to V$ if and only if $U \subset V$

Example $m = 3$

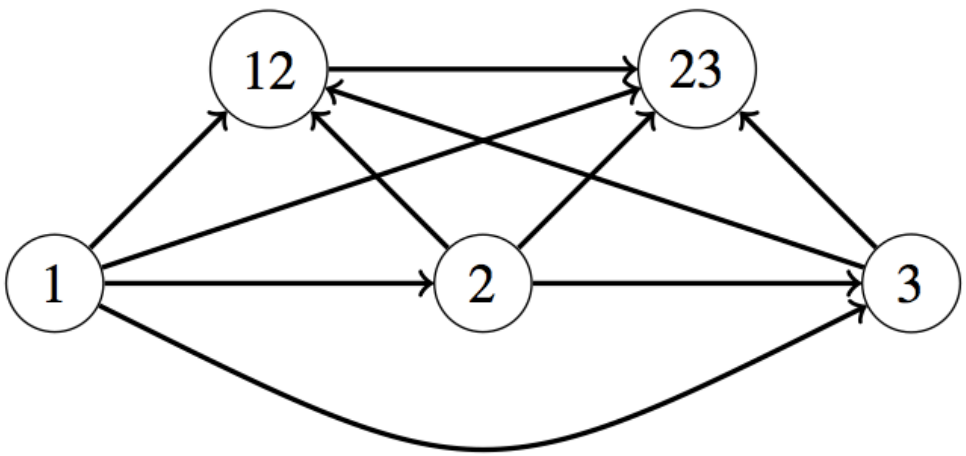


Golomb Graphs

G_m — mixed graph whose vertices are all proper consecutive subsets of $[m]$

Underlying graph is complete and $U \to V$ if and only if $U \subset V$

Example $m = 3$



Orienting a mixed graph means giving each undirected edge an orientation.

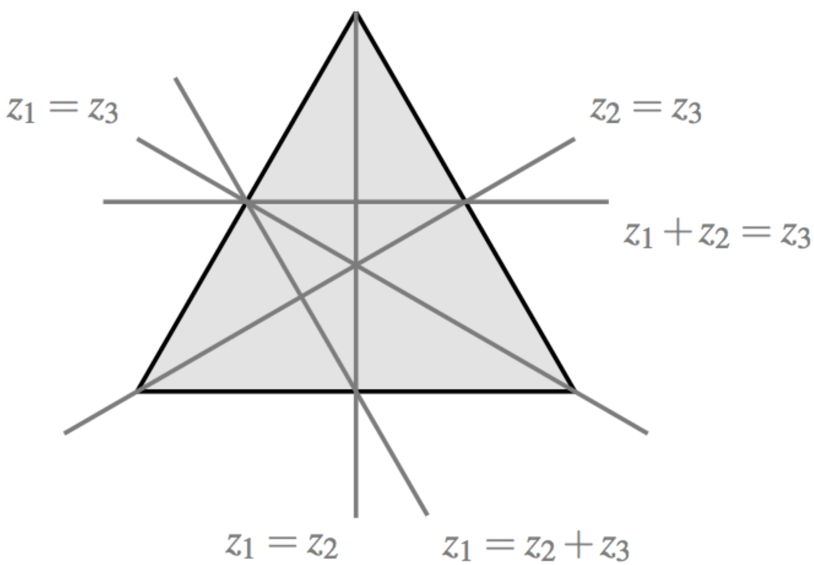
Such an orientation is acyclic if there are no coherently oriented cycles.

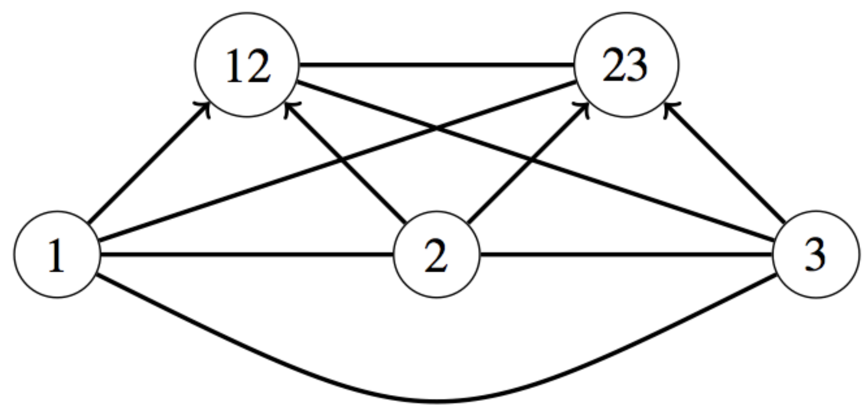
Golomb Graphs

Theorem 4 The regions of a Golomb inside-out polytope are in one-to-one correspondence with the acyclic orientations of the corresponding Golomb graph G_m that satisfy the relation

$$A \to B \qquad \Longleftrightarrow \qquad U \to V \qquad\qquad (\star)$$

for all proper consecutive subsets A and B of $[m]$ of the form $A = U \cup W$ and $B = V \cup W$ for some nonempty disjoint sets U, V, W.





Golomb Graphs

Theorem 4 The regions of a Golomb inside-out polytope are in one-to-one correspondence with the acyclic orientations of the corresponding Golomb graph G_m that satisfy the relation

$$A \to B \qquad \Longleftrightarrow \qquad U \to V \qquad\qquad (\star)$$

for all proper consecutive subsets A and B of $[m]$ of the form $A = U \cup W$ and $B = V \cup W$ for some nonempty disjoint sets U, V, W.

Corollary $(-1)^{m-1} g_m(-t)$ equals the number of rulers in $\mathbb{Z}^m_{\geq 0}$ of length t each counted with multiplicity equal to the number of compatible acyclic orientations of G_m that satisfy $(\star)$. Furthermore, $(-1)^{m-1} g_m(0)$ equals the number of acyclic orientations of G_m that satisfy $(\star)$.

Chromatic Polynomials of Graphs

$G = (V, E)$ — graph (without loops)

k-coloring of G — mapping $\boldsymbol{x} \in \{1, 2, \ldots, k\}^V$

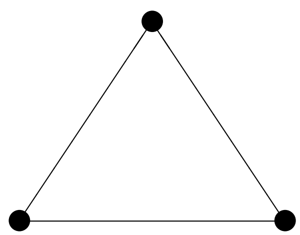
Chromatic Polynomials of Graphs

$G = (V, E)$ — graph (without loops)

Proper k-coloring of G — $\boldsymbol{x} \in \{1, 2, \ldots, k\}^V$ such that $x_i \neq x_j$ if $ij \in E$

$\chi_G(k) := \#\,(\text{proper } k\text{-colorings of } G)$

Example



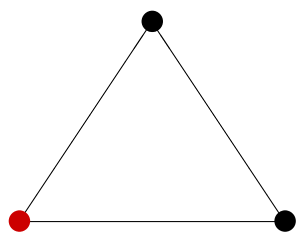
Chromatic Polynomials of Graphs

$G = (V, E)$ — graph (without loops)

Proper k-coloring of G — $\boldsymbol{x} \in \{1, 2, \ldots, k\}^V$ such that $x_i \neq x_j$ if $ij \in E$

$\chi_G(k) := \#\,($proper k-colorings of $G)$

Example



$\chi_{K_3}(k) = k \cdots$

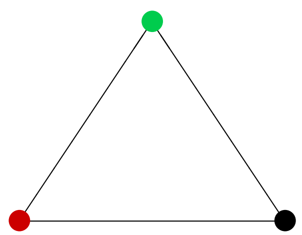
Chromatic Polynomials of Graphs

$G = (V, E)$ — graph (without loops)

Proper k-coloring of G — $\boldsymbol{x} \in \{1, 2, \ldots, k\}^V$ such that $x_i \neq x_j$ if $ij \in E$

$\chi_G(k) := \#$ (proper k-colorings of G)

Example



$\chi_{K_3}(k) = k(k-1) \cdots$

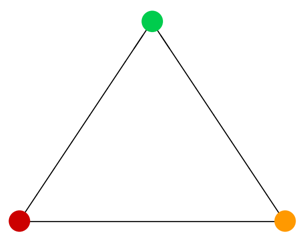
Chromatic Polynomials of Graphs

$G = (V, E)$ — graph (without loops)

Proper k-coloring of G — $\boldsymbol{x} \in \{1, 2, \ldots, k\}^V$ such that $x_i \neq x_j$ if $ij \in E$

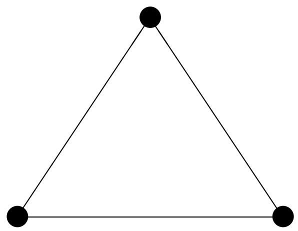
$\chi_G(k) := \#$ (proper k-colorings of G)

Example



$$\chi_{K_3}(k) = k(k-1)(k-2)$$

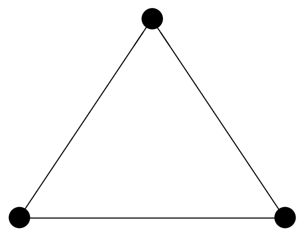
Chromatic Polynomials of Graphs



$$\chi_{K_3}(k) = k(k-1)(k-2)$$

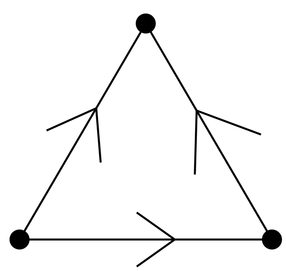
Theorem (Birkhoff 1912, Whitney 1932) $\chi_G(k)$ is a polynomial in k.

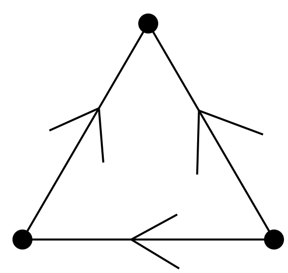
Chromatic Polynomials of Graphs



$$\chi_{K_3}(k) = k(k-1)(k-2)$$

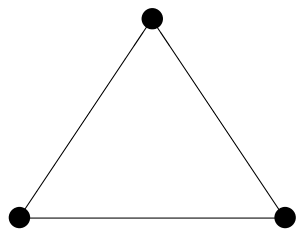
Theorem (Birkhoff 1912, Whitney 1932) $\chi_G(k)$ is a polynomial in k.





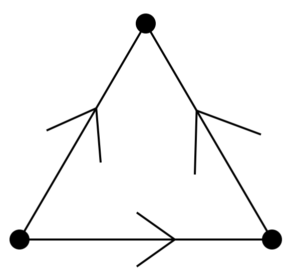
$|\chi_{K_3}(-1)| = 6$ counts the number of acyclic orientations of K_3.

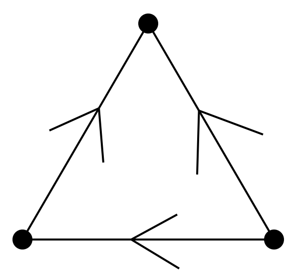
Chromatic Polynomials of Graphs



$\chi_{K_3}(k) = k(k-1)(k-2)$

Theorem (Birkhoff 1912, Whitney 1932) $\chi_G(k)$ is a polynomial in k.





$|\chi_{K_3}(-1)| = 6$ counts the number of acyclic orientations of K_3.

Theorem (Stanley 1973) $(-1)^{|V|}\chi_G(-k)$ equals the number of pairs $(\alpha, \boldsymbol{x})$ consisting of an acyclic orientation α of G and a compatible k-coloring $\boldsymbol{x}$. In particular, $(-1)^{|V|}\chi_G(-1)$ equals the number of acyclic orientations of G.

Chromatic Polynomials of Mixed Graphs

$G = (V, E, A)$ where E contains the undirected edges and A the directed edges.

Proper k-coloring of G — $\boldsymbol{x} \in \{1, 2, \ldots, k\}^V$ such that $x_i \neq x_j$ if $ij \in E$ and $x_i < x_j$ if $ij \in A$

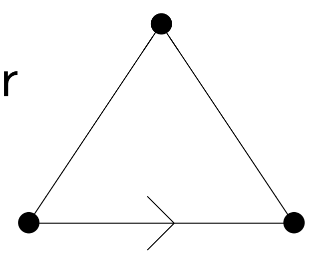
Chromatic Polynomials of Mixed Graphs

$G = (V, E, A)$ where E contains the undirected edges and A the directed edges.

Proper k-coloring of G — $\boldsymbol{x} \in \{1, 2, \ldots, k\}^V$ such that $x_i \neq x_j$ if $ij \in E$ and $x_i < x_j$ if $ij \in A$

$\chi_G(k) := \#\,(\text{proper } k\text{-colorings of } G)$

Fun Exercise Compute $\chi_G(k)$ for



Theorem (Sotskov–Tanaev–Werner 2002) $\chi_G(k)$ is a polynomial in k.

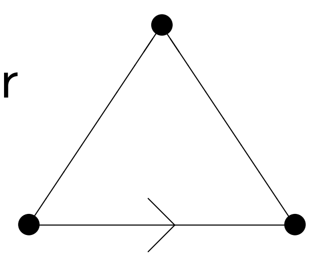
Chromatic Polynomials of Mixed Graphs

$G = (V, E, A)$ where E contains the undirected edges and A the directed edges.

Proper k-coloring of G — $\boldsymbol{x} \in \{1, 2, \ldots, k\}^V$ such that $x_i \neq x_j$ if $ij \in E$ and $x_i < x_j$ if $ij \in A$

$\chi_G(k) := \#\,(\text{proper } k\text{-colorings of } G)$

Fun Exercise Compute $\chi_G(k)$ for



Theorem (Sotskov–Tanaev–Werner 2002) $\chi_G(k)$ is a polynomial in k.

Theorem 5 $(-1)^{|V|}\chi_G(-k)$ equals the number of pairs $(\alpha, \boldsymbol{x})$ consisting of an acyclic orientation α of G and a compatible k-coloring $\boldsymbol{x}$. In particular, $(-1)^{|V|}\chi_G(-1)$ equals the number of acyclic orientations of G.

Open Problems

- Optimal Golomb rulers
- Smallest positive integer root of $g_m(t)$
- Compute $g_m(t)$ . . . period? constant term?
- Mixed chromatic polynomials