GEOMETRIC PROOFS OF POLYNOMIAL RECIPROCITY LAWS OF CARLITZ, BERNDT, AND DIETER

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Dedicated to Iekata Shiokawa on the occasion of his 65th birthday and retirement.

Abstract. We study higher-dimensional analogs of the *Carlitz polynomials*

$$\mathrm{c}\,(u,v;a,b) := \sum_{k=1}^{a-1} u^{k-1} v^{\left\lfloor \frac{kb}{a} \right\rfloor},$$

where u and v are indeterminants, a and b are positive integers, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. These polynomials satisfy a reciprocity law, from which one easily concludes many classical reciprocity theorems for the Dedekind sum and its generalizations, most notably by Hardy and Berndt–Dieter. We give new proofs of some general reciprocity theorems of Berndt and Dieter, using lattice points in polyhedral regions.

1. Introduction

Our goal is to study higher-dimensional analogs of the polynomials

$$\mathrm{c}\,(u,v;a,b) := \sum_{k=1}^{a-1} u^{k-1} v^{\left\lfloor \frac{kb}{a} \right\rfloor}, \tag{1}$$

where u and v are indeterminants, a and b are positive integers, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. Polynomials of the form of $\mathrm{c}\,(u,v;a,b)$ were introduced by Carlitz [5], who proved the following *reciprocity law*:

Theorem 1.1 (Carlitz). *If a and b are relatively prime positive integers, then*

$$(u-1)\,\mathrm{c}(u,v;a,b) + (v-1)\,\mathrm{c}(v,u;b,a) = u^{a-1}v^{b-1} - 1\ .$$

(Carlitz's reciprocity theorem stated in [5] is of a slightly different form, which is easily seen to be equivalent to Theorem 1.1, which in this form was first published by Berndt and Dieter [3].)

Theorem 1.1 implies the classical reciprocity law for the *Dedekind sum* [6, 11]

$$\mathrm{s}\,(a,b) := \sum_{k=1}^{b-1} \left(\!\!\left(\frac{ka}{b}\right)\!\!\right)\left(\!\!\left(\frac{k}{b}\right)\!\!\right), \qquad \text{where} \qquad ((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

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By applying the operators $u\frac{\partial}{\partial u}$ and $v\frac{\partial}{\partial v}$ to the identity of Theorem 1.1 and subsequently setting $u = v = 1$, one obtains (after some easy algebraic manipulation) the reciprocity law

$$\mathrm{s}\,(a, b) + \mathrm{s}\,(b, a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right), \tag{2}$$

arguably the most important property of $\mathrm{s}\,(a, b)$. This reciprocity law is the essential ingredient to the transformation formula for the Dedekind η-function [6], and (2) implies Gauss's theorem of quadratic reciprocity [11]. Together with the modular property $\mathrm{s}\,(a, b) = \mathrm{s}\,(a \bmod b, b)$, Dedekind's reciprocity law (2) also implies that $\mathrm{s}\,(a, b)$ can be efficiently computed by means of a Euclidean-type algorithm.

Other Dedekind-like sums, due to Hardy [8] and Berndt [2] and appearing in transformation formulas for theta functions, are

$$S(h,k) = \sum_{j=1}^{k-1} (-1)^{j+1+\left\lfloor \frac{hj}{k} \right\rfloor} \qquad s_1(h,k) = \sum_{j=1}^{k} (-1)^{\left\lfloor \frac{hj}{k} \right\rfloor} \left(\left(\frac{j}{k}\right)\right)$$

$$s_2(h,k) = \sum_{j=1}^{k} (-1)^{j} \left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{hj}{k}\right)\right) \qquad s_3(h,k) = \sum_{j=1}^{k} (-1)^{j} \left(\left(\frac{hj}{k}\right)\right)$$

$$s_4(h,k) = \sum_{j=1}^{k-1} (-1)^{\left\lfloor \frac{hj}{k} \right\rfloor} \qquad s_5(h,k) = \sum_{j=1}^{k} (-1)^{j+\left\lfloor \frac{hj}{k} \right\rfloor} \left(\left(\frac{j}{k}\right)\right).$$

These *Hardy–Berndt sums* also satisfy certain reciprocity relations, originally proved by Berndt [2]. (Hardy stated in [8] some of the reciprocity laws without proof.) Apostol–Vu [1] and Berndt–Dieter [3] showed how one can use Carlitz's Theorem 1.1 to give simple proofs of the reciprocity relations of the Hardy–Berndt sums.

Our purpose in this paper is to study the following higher-dimensional generalization of the polynomials of the form (1).

Definition. The *Carlitz polynomial* $\mathrm{c}\,(u_1, u_2, \dots, u_n; a_1, a_2, \dots, a_n)$, where $u_1, u_2, \dots, u_n$ are indeterminants and $a_1, a_2, \dots, a_n$ are positive integers, is defined as the polynomial

$$\mathrm{c}\,(u_1, u_2, \dots, u_n; a_1, a_2, \dots, a_n) := \sum_{k=1}^{a_1-1} u_1^{k-1} u_2^{\left\lfloor \frac{ka_2}{a_1} \right\rfloor} u_3^{\left\lfloor \frac{ka_3}{a_1} \right\rfloor} \cdots u_n^{\left\lfloor \frac{ka_n}{a_1} \right\rfloor}.$$

The main contribution of this note is a geometric proof of the following reciprocity law, based on a double lattice-point enumeration in certain polytopes (Section 2).

Theorem 1.2 (Berndt–Dieter)**.** *If $a_1, a_2, \dots, a_n$ are pairwise relatively prime positive integers, then*

$$(u_1 - 1)\,\mathrm{c}\,(u_1, u_2, \dots, u_n; a_1, a_2, \dots, a_n) + (u_2 - 1)\,\mathrm{c}\,(u_2, u_3, \dots, u_n, u_1; a_2, a_3 \dots, a_n, a_1) + \cdots$$
$$+ (u_n - 1)\,\mathrm{c}\,(u_n, u_1, \dots, u_{n-1}; a_n, a_1, \dots, a_{n-1}) = u_1^{a_1-1} u_2^{a_2-1} \cdots u_n^{a_n-1} - 1\ .$$

Carlitz's Theorem 1.1 is the case $n = 2$ of this identity. The general case of Theorem 1.2 was given by Berndt and Dieter [3, Theorem 5.1]. In fact, they gave a slightly more general polynomial reciprocity law, which we could also prove with our geometric methods; however, for the sake of simplicity of exposition, we chose to concentrate on Theorem 1.2. Pettet and Rao [9] used the case $n = 3$ of Theorem 1.2 to give simple proofs of three-term reciprocity laws of Goldberg for the Hardy–Berndt sums [7] and of Rademacher for the Dedekind sum [10].

Just like Theorem 1.1 implies reciprocity laws for the Dedekind and Hardy–Berndt sums, Theorem 1.2 has numerous corollaries, some of which we describe in Section 3.

2. Proof of the Reciprocity Law for Carlitz Polynomials

For ease of exposition, we first outline our approach to proving Theorem 1.2 for the two-term case, that is, Theorem 1.1. After proving this case in detail, we will discuss how to show the general case.

Proof of Theorem 1.1. We rewrite the sum on the left-hand side of the identity as

$$(u-1)\,\mathrm{c}(u,v;a,b)+(v-1)\,\mathrm{c}(v,u;b,a)=\sum_{k=1}^{a-1}u^k v^{\left\lfloor\frac{kb}{a}\right\rfloor}+\sum_{j=1}^{b-1}v^j u^{\left\lfloor\frac{ja}{b}\right\rfloor}-\sum_{k=1}^{a-1}u^{k-1}v^{\left\lfloor\frac{kb}{a}\right\rfloor}-\sum_{j=1}^{b-1}v^{j-1}u^{\left\lfloor\frac{ja}{b}\right\rfloor}$$

and pair up the monomials with positive and with negative sign. More precisely, we collect the exponents (x,y) of the monomials $u^x v^y$ in the four sets

$$\begin{aligned}
A_+ &:= \left\{\left(k, \left\lfloor\tfrac{kb}{a}\right\rfloor\right) :\ 1\le k\le a-1\right\},\\
B_+ &:= \left\{\left(\left\lfloor\tfrac{ja}{b}\right\rfloor, j\right) :\ 1\le j\le b-1\right\},\\
A_- &:= \left\{\left(k-1, \left\lfloor\tfrac{kb}{a}\right\rfloor\right) :\ 1\le k\le a-1\right\},\\
B_- &:= \left\{\left(\left\lfloor\tfrac{ja}{b}\right\rfloor, j-1\right) :\ 1\le j\le b-1\right\}.
\end{aligned}$$

We will show that $A_+\cup B_+$ and $A_-\cup B_-$ are disjoint unions that differ only in the elements $(0,0)$ and $(a-1,b-1)$, which give rise the right-hand side of the identity in Theorem 1.1.

There is a simple geometric picture underlying our proof, indicated in Figure 1. Namely, we consider all $\mathbb{Z}^2$-lattice points in the half-open polyhedron

$$P := \left\{(x,y)\in\mathbb{R}^2 :\ \begin{array}{c}0\le x\le a-1\\ 0\le y\le b-1\end{array}\ ,\ \begin{array}{c}x> y\frac{a}{b}-1\\ y> x\frac{b}{a}-1\end{array}\right\},$$

which is sketched in Figure 1. The sets $A_\pm, B_\pm$ simply contain lattice points in certain parts of this polyhedron, and the Carlitz polynomials encode these lattice points, each point (k,j) as the monomial $u^k v^j$. From Figure 1 one can read off the disjoint unions $A_+\cup B_+ = \left(P\cap\mathbb{Z}^2\right)\setminus\{(0,0)\}$ and $A_-\cup B_- = \left(P\cap\mathbb{Z}^2\right)\setminus\{(a-1,b-1)\}$, which is our claim and gives Theorem 1.1.

Now for the non-picturesque proof. We start by proving that A_+ and B_+ are disjoint; the fact that A_- and B_- are disjoint follows analogously. Assume that $k=\left\lfloor\frac{ja}{b}\right\rfloor$ and $j=\left\lfloor\frac{kb}{a}\right\rfloor$; then

$$k\le j\tfrac{a}{b}\le \tfrac{kb}{a}\,\tfrac{a}{b}=k$$

and hence $kb=ja$. But since a and b are coprime, this can only happen if $a|k$; however, $1\le k\le a-1$.

Next, let

$$S := P\cap\mathbb{Z}^2 = \left\{(x,y)\in\mathbb{Z}^2 :\ \begin{array}{c}0\le x\le a-1\\ 0\le y\le b-1\end{array}\ ,\ \begin{array}{c}x> y\frac{a}{b}-1\\ y> x\frac{b}{a}-1\end{array}\right\}.$$

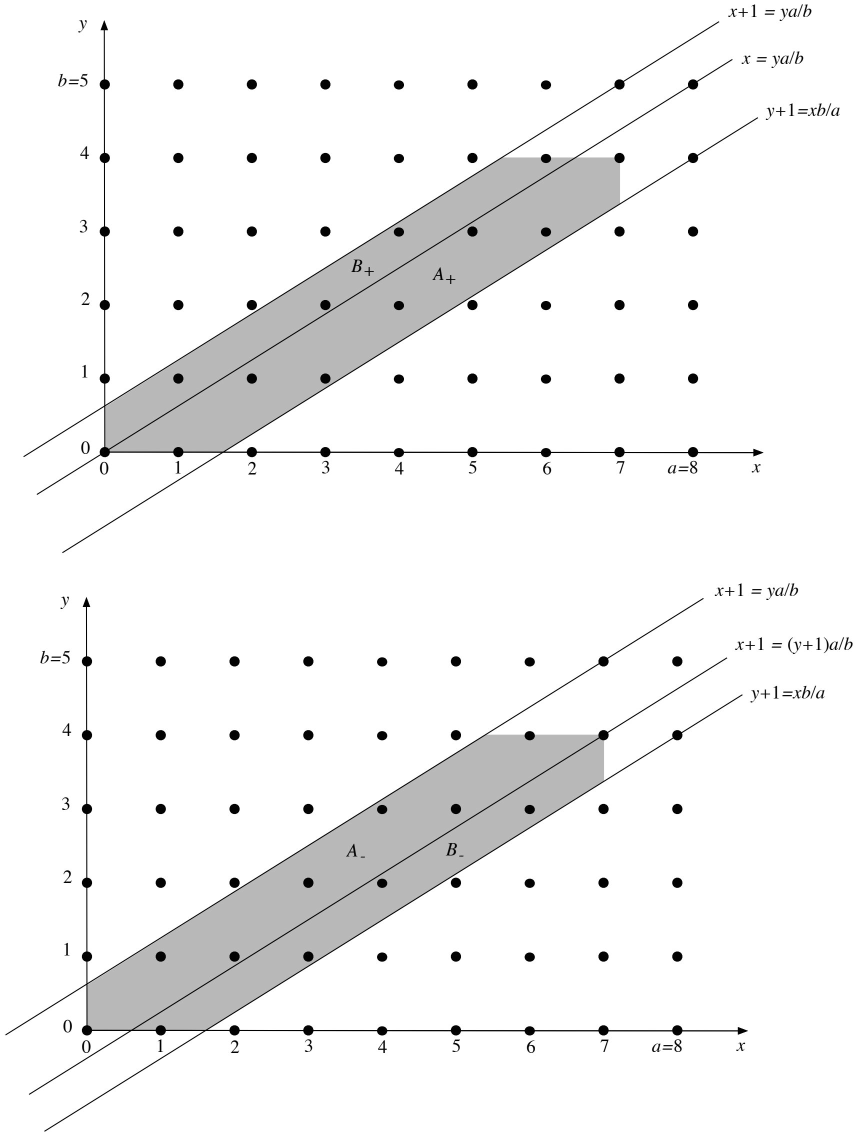

FIGURE 1. The geometry behind the proof.

We claim that $A_+ \cup B_+ = S \setminus \{(0,0)\}$ and $A_- \cup B_- = S \setminus \{(a-1, b-1)\}$, which proves Theorem 1.1. To prove our claim, we first note that $A_\pm \subseteq S$ (and similarly $B_\pm \subseteq S$) because

$$\left\lfloor \frac{kb}{a} \right\rfloor > k\frac{b}{a} - 1 \ ,$$

and that $(0,0) \notin A_+ \cup B_+$ and $(a-1, b-1) \notin A_- \cup B_-$. It remains to show

(i) $S \setminus \{(0,0)\} \subseteq A_+ \cup B_+$;
(ii) $S \setminus \{(a-1, b-1)\} \subseteq A_- \cup B_-$.

To prove (i), assume $(x,y) \in S \setminus \{(0,0)\}$. By construction of S, $y\frac{a}{b} - 1 < x$. If $x \le \frac{ya}{b}$, then $x = \left\lfloor \frac{ya}{b} \right\rfloor$ and $(x,y) \in B_+$. Otherwise, $x > \frac{ya}{b}$, so that $y < \frac{xb}{a}$ and, since $x\frac{b}{a} - 1 < y$, we conclude that $y = \left\lfloor \frac{xb}{a} \right\rfloor$ and $(x,y) \in A_+$.

To prove (ii), suppose $(x,y) \in S \setminus \{(a-1,b-1)\}$, and let

$$k := x+1 \qquad \text{and} \qquad j := y+1 \; .$$

By construction of S, $x < (y+1)\frac{a}{b} = j\frac{a}{b}$. If $j\frac{a}{b} - 1 < x$, then $x = \left\lfloor \frac{ja}{b} \right\rfloor$, that is, $(x,y) = \left(\left\lfloor \frac{ja}{b} \right\rfloor, j-1 \right) \in B_-$. Otherwise, $j\frac{a}{b} - 1 \ge x$, and hence $k \le j\frac{a}{b} = (y+1)\frac{a}{b}$, that is, $\frac{kb}{a} - 1 \le y$. Since $\frac{kb}{a} \notin \mathbb{Z}$ (note that $k < a$—if $k = a$ then $j = b$, a case that we excluded), we can make this inequality strict: $\frac{kb}{a} - 1 < y$. Furthermore, by construction of S, $y < (x+1)\frac{b}{a} = \frac{kb}{a}$, whence we have $y = \left\lfloor \frac{kb}{a} \right\rfloor$, that is, $(x,y) = \left(k-1, \left\lfloor \frac{kb}{a} \right\rfloor\right) \in A_-$. □

The proof for $n = 2$ contains already the essential ingredients needed for the general Theorem 1.2. For general n, again we will construct a (half-open) polytope, and the Carlitz polynomials in the reciprocity law encode different sets of lattice points in this polytope.

Proof of Theorem 1.2. As in the above proof for $n = 2$, we group the terms on the left-hand side of the identity according to sign as

$$\sum_{k=1}^{a_1-1} u_1^k u_2^{\left\lfloor \frac{ka_2}{a_1} \right\rfloor} u_3^{\left\lfloor \frac{ka_3}{a_1} \right\rfloor} \cdots u_n^{\left\lfloor \frac{ka_n}{a_1} \right\rfloor} + \cdots + \sum_{k=1}^{a_n-1} u_n^k u_1^{\left\lfloor \frac{ka_1}{a_n} \right\rfloor} u_2^{\left\lfloor \frac{ka_2}{a_n} \right\rfloor} \cdots u_{n-1}^{\left\lfloor \frac{ka_{n-1}}{a_n} \right\rfloor}$$
$$- \sum_{k=1}^{a_1-1} u_1^{k-1} u_2^{\left\lfloor \frac{ka_2}{a_1} \right\rfloor} u_3^{\left\lfloor \frac{ka_3}{a_1} \right\rfloor} \cdots u_n^{\left\lfloor \frac{ka_n}{a_1} \right\rfloor} - \cdots - \sum_{k=1}^{a_n-1} u_n^{k-1} u_1^{\left\lfloor \frac{ka_1}{a_n} \right\rfloor} u_2^{\left\lfloor \frac{ka_2}{a_n} \right\rfloor} \cdots u_{n-1}^{\left\lfloor \frac{ka_{n-1}}{a_n} \right\rfloor}$$

and collect the respective exponents in the sets

$$\begin{aligned}
A_+^1 &:= \left\{ \left(k, \left\lfloor \frac{ka_2}{a_1} \right\rfloor, \left\lfloor \frac{ka_3}{a_1} \right\rfloor, \dots, \left\lfloor \frac{ka_n}{a_1} \right\rfloor \right) : \ 1 \le k \le a_1 - 1 \right\} \\
A_+^2 &:= \left\{ \left(\left\lfloor \frac{ka_1}{a_2} \right\rfloor, k, \left\lfloor \frac{ka_3}{a_2} \right\rfloor, \dots, \left\lfloor \frac{ka_n}{a_2} \right\rfloor \right) : \ 1 \le k \le a_2 - 1 \right\} \\
&\ \vdots \\
A_+^n &:= \left\{ \left(\left\lfloor \frac{ka_1}{a_n} \right\rfloor, \left\lfloor \frac{ka_2}{a_n} \right\rfloor, \dots, \left\lfloor \frac{ka_{n-1}}{a_n} \right\rfloor, k \right) : \ 1 \le k \le a_n - 1 \right\} \\
A_-^1 &:= \left\{ \left(k-1, \left\lfloor \frac{ka_2}{a_1} \right\rfloor, \left\lfloor \frac{ka_3}{a_1} \right\rfloor, \dots, \left\lfloor \frac{ka_n}{a_1} \right\rfloor \right) : \ 1 \le k \le a_1 - 1 \right\} \\
A_-^2 &:= \left\{ \left(\left\lfloor \frac{ka_1}{a_2} \right\rfloor, k-1, \left\lfloor \frac{ka_3}{a_2} \right\rfloor, \dots, \left\lfloor \frac{ka_n}{a_2} \right\rfloor \right) : \ 1 \le k \le a_2 - 1 \right\} \\
&\ \vdots \\
A_-^n &:= \left\{ \left(\left\lfloor \frac{ka_1}{a_n} \right\rfloor, \left\lfloor \frac{ka_2}{a_n} \right\rfloor, \dots, \left\lfloor \frac{ka_{n-1}}{a_n} \right\rfloor, k-1 \right) : \ 1 \le k \le a_n - 1 \right\} .
\end{aligned}$$

Define

$$S := \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n : \; 0 \le x_j \le a_j - 1, \, x_k > x_j \frac{a_k}{a_j} - 1 \text{ for all } 1 \le j, k \le n \right\};$$

we will show that $\bigcup_{j=1}^n A_+^j = S \setminus \{(0,0,\ldots,0)\}$ and $\bigcup_{j=1}^n A_-^j = S \setminus \{(a_1-1, a_2-1, \ldots, a_n-1)\}$. Both unions are pairwise disjoint, which follows exactly like in our above proof for $n = 2$. Also just like in the case $n = 2$, one easily sees that $A_\pm^j \subseteq S$ and that $(0,0,\ldots,0) \notin \bigcup_{j=1}^n A_+^j$ and $(a_1-1, a_2-1, \ldots, a_n-1) \notin \bigcup_{j=1}^n A_-^j$, so that it remains to prove:

(i) $S \setminus \{(0,0,\ldots,0)\} \subseteq \bigcup_{j=1}^n A_+^j$;

(ii) $S \setminus \{(a_1-1, a_2-1, \ldots, a_n-1)\} \subseteq \bigcup_{j=1}^n A_-^j$.

To prove (i), assume $x := (x_1, x_2, \ldots, x_n) \in S \setminus \{(0,0,\ldots,0)\}$. Let $\frac{x_k}{a_k}$ be the maximum of $\left\{ \frac{x_1}{a_1}, \frac{x_2}{a_2}, \ldots, \frac{x_n}{a_n} \right\}$. Hence $x_j \le x_k \frac{a_j}{a_k}$ for all j, and, by construction of S, $x_k \frac{a_j}{a_k} - 1 < x_j$ for all $j \ne k$. That is, $x_j = \left\lfloor \frac{x_k a_j}{a_k} \right\rfloor$ for all $j \ne k$, and $x \in A_+^k$.

To prove (ii), assume $x := (x_1, x_2, \ldots, x_n) \in S \setminus \{(a_1-1, a_2-1, \ldots, a_n-1)\}$. Let $\frac{x_k+1}{a_k}$ be the minimum of $\left\{ \frac{x_1+1}{a_1}, \frac{x_2+1}{a_2}, \ldots, \frac{x_n+1}{a_n} \right\}$. Hence $(x_k+1) \frac{a_j}{a_k} - 1 \le x_j$ for all $j \ne k$, which can be replaced by a strict inequality since $(x_k+1) \frac{a_j}{a_k} \notin \mathbb{Z}$ (note that $x_k < a_k - 1$). Futhermore, by construction of S, $x_j < (x_k+1) \frac{a_j}{a_k}$ for all $j \ne k$. That is, $x_j = \left\lfloor \frac{(x_k+1)a_j}{a_k} \right\rfloor$ for all $j \ne k$, and $x \in A_-^k$. □

3. A Few Applications

The archetype of corollaries that are immediately implied by Theorem 1.2 is the following identity.

Corollary 3.1. *If $a_1, a_2, \ldots, a_n$ are pairwise relatively prime positive integers, then*

$$\sum_{k=1}^{a_1-1} \left\lfloor \frac{ka_2}{a_1} \right\rfloor \left\lfloor \frac{ka_3}{a_1} \right\rfloor \cdots \left\lfloor \frac{ka_n}{a_1} \right\rfloor + \cdots + \sum_{k=1}^{a_n-1} \left\lfloor \frac{ka_1}{a_n} \right\rfloor \left\lfloor \frac{ka_2}{a_n} \right\rfloor \cdots \left\lfloor \frac{ka_{n-1}}{a_n} \right\rfloor = (a_1-1)(a_2-1)\cdots(a_n-1).$$

Proof. Apply the operator $\left(u_1 \left(\frac{\partial}{\partial u_1}\right)\right) \left(u_2 \left(\frac{\partial}{\partial u_2}\right)\right) \cdots \left(u_n \left(\frac{\partial}{\partial u_n}\right)\right)$ to both sides of the identity of Theorem 1.2 and set $u_1 = u_2 = ... = u_n = 1$. □

The case $n = 3$ of this identity yields, after replacing the greatest integer function with $((\ldots))$ and doing a little arithmetic, the following three-term generalization of Dedekind's reciprocity law (2) due to Rademacher [10]:

Corollary 3.2 (Rademacher)**.** *If a, b, and c are pairwise relatively prime positive integers, then*

$$\sum_{k=1}^{a-1} \left(\left(\frac{kb}{a}\right)\right) \left(\left(\frac{kc}{a}\right)\right) + \sum_{k=1}^{b-1} \left(\left(\frac{kc}{b}\right)\right) \left(\left(\frac{ka}{b}\right)\right) + \sum_{k=1}^{c-1} \left(\left(\frac{ka}{c}\right)\right) \left(\left(\frac{kb}{c}\right)\right)$$
$$= -\frac{1}{4} + \frac{1}{12} \left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \right).$$

Dedekind's reciprocity law (2) follows from Corollary 3.2 by setting $c = 1$.

It is well known (see, e.g., [11]) that the Dedekind sum has the trigonometric representation

$$\mathrm{s}\,(a,b) = \frac{1}{4b}\sum_{k=1}^{b-1} \cot\frac{\pi k a}{b}\cot\frac{\pi k}{b} \ ,$$

and hence Dedekind's reciprocity law and its various generalizations gives rise to many, sometimes curiously looking, trigonometric identities (see, e.g., [4]).

We mentioned already in the introduction that the following reciprocity laws of the Hardy–Berndt sums [2] are straightforward consequences of Carlitz's Theorem 1.1, as shown in [1, 3], so we will not repeat their proofs here.

Corollary 3.3 (Berndt)**.** *Suppose a and b are relatively prime positive integers. If $a+b$ is odd then*

$$S(a,b) + S(b,a) = 1 \ .$$

If a and b are odd then

$$s_5(a,b) + s_5(b,a) = \frac{1}{2} - \frac{1}{2ab} \ .$$

If a is even then

$$s_1(a,b) - 2\,s_2(b,a) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{ab} + \frac{a}{b}\right).$$

If b is odd then

$$2\,s_3(a,b) - s_4(b,a) = 1 - \frac{a}{b} \ .$$

Instead, we give a higher-dimensional analog of the first Hardy–Berndt sum

$$S(a,b) = \sum_{j=1}^{b-1} (-1)^{j+1+\left\lfloor \frac{ja}{b} \right\rfloor} = \mathrm{c}\,(-1,-1;b,a)\,,$$

namely the *higher-dimensional Hardy–Berndt sum*

$$\mathrm{hb}\,(a_1;a_2,\ldots,a_n) := \sum_{j=1}^{a_1-1} (-1)^{j+1+\left\lfloor \frac{ja_2}{a_1} \right\rfloor + \cdots + \left\lfloor \frac{ja_n}{a_1} \right\rfloor}.$$

It satisfies the following reciprocity law.

Corollary 3.4. *If $a_1, a_2, \ldots, a_n$ are pairwise relatively prime positive integers, then*

$$\mathrm{hb}\,(a_1;a_2,\ldots,a_n) + \mathrm{hb}\,(a_2;a_3,\ldots,a_n,a_1) + \cdots + \mathrm{hb}\,(a_n;a_1,\ldots,a_{n-1}) = \frac{1-(-1)^{a_1+a_2+\cdots+a_n+n}}{2} \ .$$

Proof. Substitute $u_1 = u_2 = \cdots = u_n = -1$ in Theorem 1.2. □

Similar higher-dimensional analogs of the other Hardy-Berndt sums and their reciprocity laws can be deduced from Theorem 1.2. We finish with the remark that we could also introduce linear shifts in the arguments of the greatest integer functions in the definition of the Carlitz sum. Our geometric proof remains essentially the same, and it yields the general polynomial reciprocity theorem of Berndt and Carlitz [3, Theorem 5.1].

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