Inside-Out Polytopes

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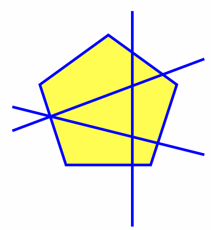

math.sfsu.edu/beck/

arXiv: math.CO/0309330 & math.CO/0309331 & math.CO/0506315

Chromatic Polynomials of Graphs

$\Gamma = (V, E)$ – graph (without loops)

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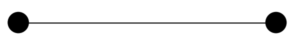
Theorem (Stanley 1973) $(-1)^{|V|}\chi_\Gamma(-k)$ equals the number of pairs (α, x) consisting of an acyclic orientation α of Γ and a compatible k-coloring. In particular, $(-1)^{|V|}\chi_\Gamma(-1)$ equals the number of acyclic orientations of Γ.

(An orientation α of Γ and a k-coloring x are compatible if $x_j \geq x_i$ whenever there is an edge oriented from i to j. An orientation is acyclic if it has no directed cycles.)

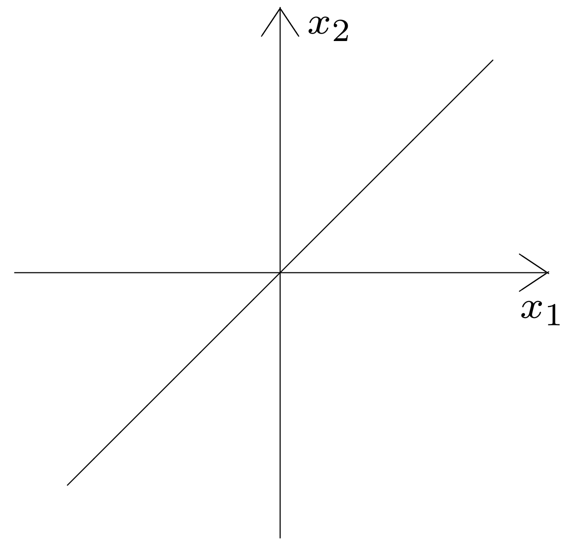
Graphical Hyperplane Arrangements

We associate with $\Gamma = (V, E)$ the hyperplane arrangement

$$\mathcal{H}_\Gamma := \{x_i = x_j : ij \in E\}$$



K_2



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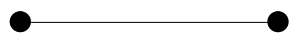

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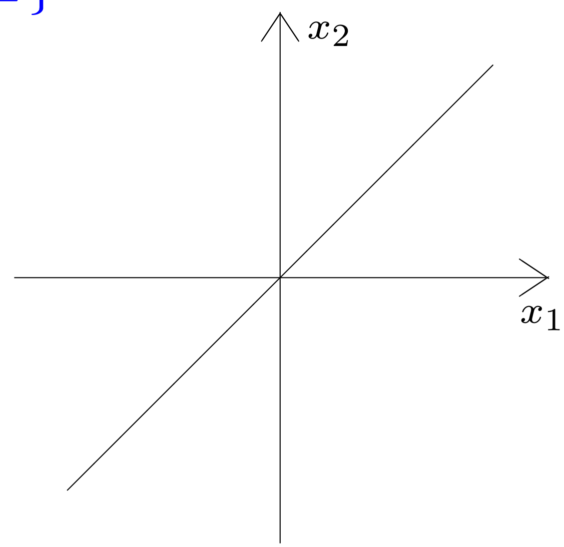

Greene's observation

region of $\mathcal{H}_\Gamma$	$\longleftrightarrow$	acyclic orientation of Γ
$x_i < x_j$	$\longleftrightarrow$	orient $ij \in E$ from i to j

Ehrhart Polynomials

$\mathcal{P} \subset \mathbb{R}^d$ – lattice polytope, i.e., the convex hull of finitely points in $\mathbb{Z}^d$

For $k \in \mathbb{Z}_{>0}$ let $\mathrm{Ehr}_{\mathcal{P}}(k) := \# \left(\mathcal{P} \cap \frac{1}{k}\mathbb{Z}^d \right) = \# \left(k\mathcal{P} \cap \mathbb{Z}^d \right)$

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(Macdonald 1971) $(-1)^{\dim \mathcal{P}} \, \mathrm{Ehr}_{\mathcal{P}}(-k) = \mathrm{Ehr}_{\mathcal{P}^\circ}(k)$, where $\mathcal{P}^\circ$ denotes the interior of $\mathcal{P}$.

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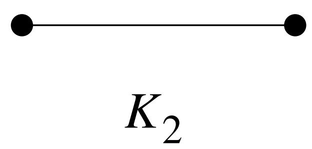
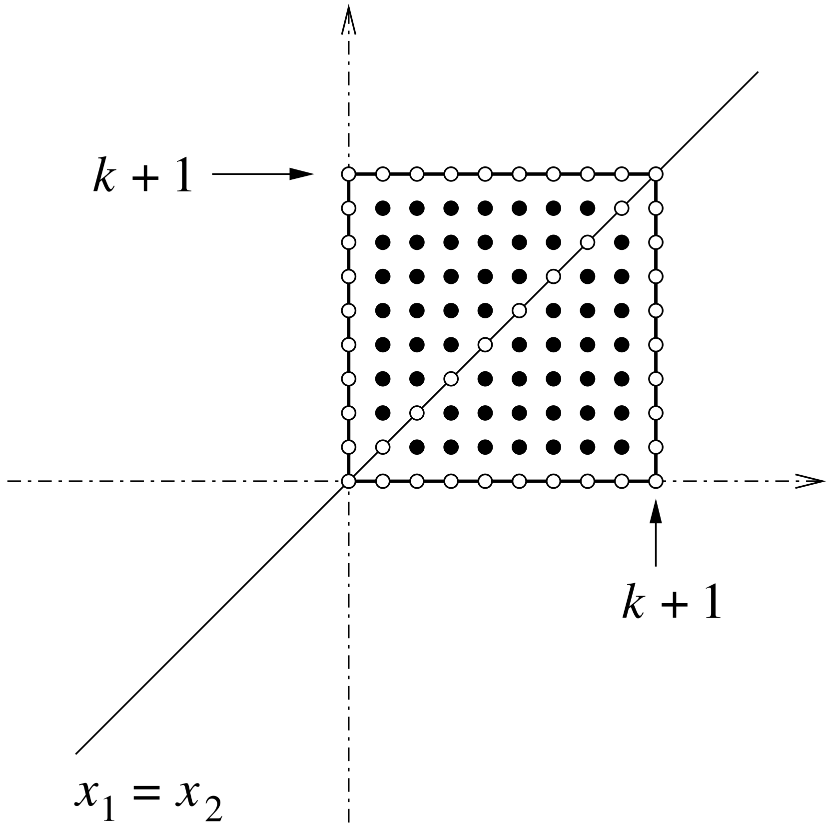
Idea

A k-coloring of Γ is an interior lattice point of $(k+1)\mathcal{P}$, where $\mathcal{P} = [0,1]^V$.

Graph Coloring a la Ehrhart

$\chi_{K_2}(k) = k(k-1)$ ...





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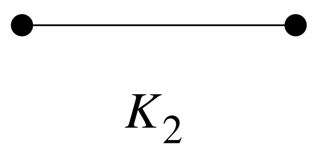

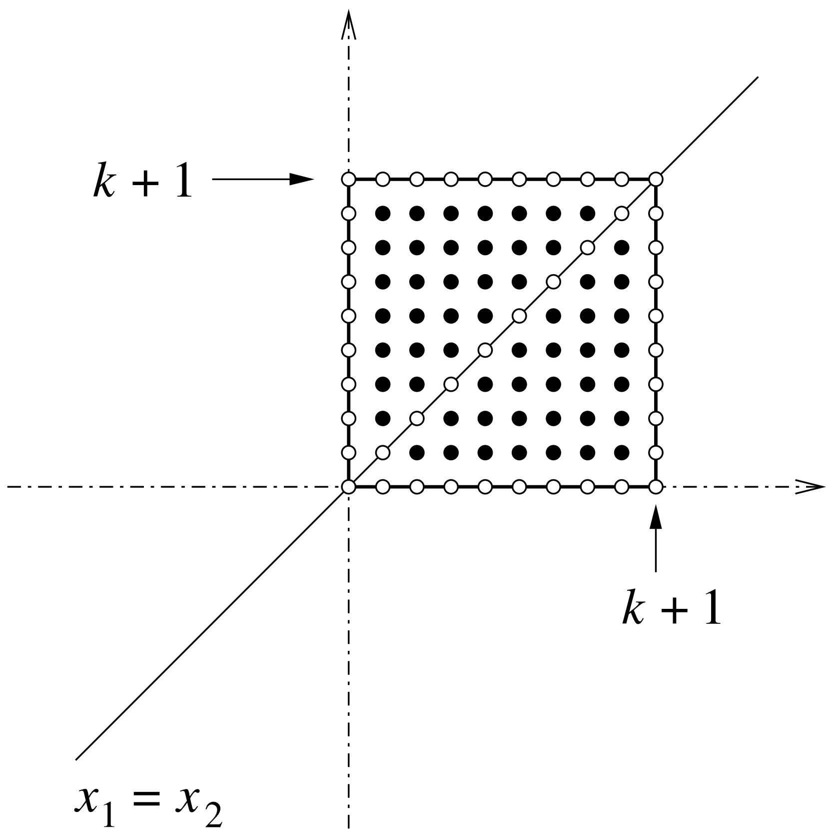

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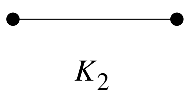
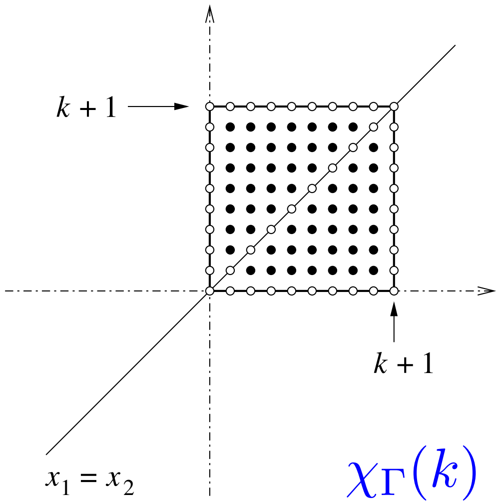
Stanley's Theorem a la Ehrhart





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Write $(0,1)^V \setminus \bigcup \mathcal{H}_\Gamma = \bigcup_j \mathcal{P}_j^\circ$, then by Ehrhart-Macdonald reciprocity

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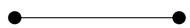

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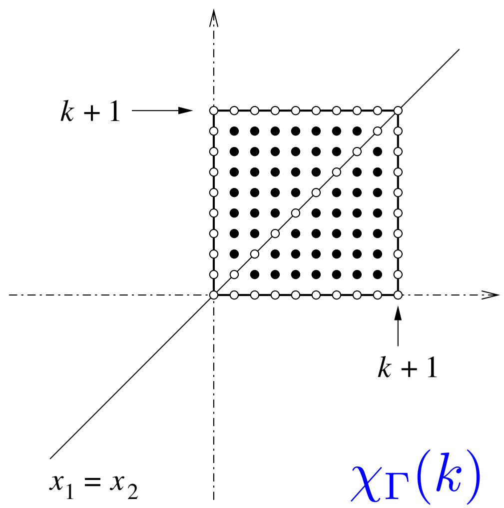

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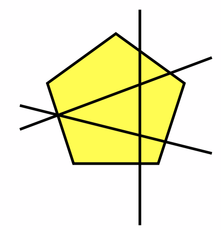
region of $\mathcal{H}_\Gamma \;\longleftrightarrow\;$ acyclic orientation of Γ

Inside-Out Counting Functions



Inside-out polytope : $(\mathcal{P}, \mathcal{H})$

Multiplicity of $x \in \mathbb{R}^d$:

$$m_{\mathcal{P},\mathcal{H}}(x) := \begin{cases} \#\text{ closed regions of } \mathcal{H} \text{ in } \mathcal{P} \text{ that contain } x & \text{if } x \in \mathcal{P}, \\ 0 & \text{if } x \notin \mathcal{P} \end{cases}$$

Closed Ehrhart quasipolynomial $E_{P,\mathcal{H}}(k) := \sum_{x \in \frac{1}{k}\mathbb{Z}^d} m_{\mathcal{P},\mathcal{H}}(x)$

Open Ehrhart quasipolynomial $E^{\circ}_{\mathcal{P},\mathcal{H}}(k) := \# \left(\frac{1}{k}\mathbb{Z}^d \cap [\mathcal{P} \setminus \bigcup \mathcal{H}] \right)$

Inside-Out Philosophy

Theorem If $(\mathcal{P}, \mathcal{H})$ is a closed, full-dimensional, rational inside-out polytope, then $E_{\mathcal{P},\mathcal{H}}(k)$ and $E^{\circ}_{\mathcal{P}^{\circ},\mathcal{H}}(k)$ are quasipolynomials in k of degree $\dim \mathcal{P}$ with leading term $\operatorname{vol} P$, and with constant term $E_{\mathcal{P},\mathcal{H}}(0)$ equal to the number of regions of $(\mathcal{P}, \mathcal{H})$. Furthermore,

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Philosophy If you have an enumeration problem that can be encoded as $E^{\circ}_{\mathcal{P}^\circ,\mathcal{H}}(k) = \# \left(\frac{1}{k}\mathbb{Z}^d \cap [\mathcal{P}^\circ \setminus \bigcup \mathcal{H}] \right)$ for some inside-out polytope $(\mathcal{P}, \mathcal{H})$ and you have a combinatorial interpretation for the multiplicities $m_{\mathcal{P},\mathcal{H}}(x)$, then you'll have a combinatorial reciprocity theorem for $E^{\circ}_{\mathcal{P}^\circ,\mathcal{H}}(k)$.

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Theorem $(\mathcal{P}, \mathcal{H})$ is a closed, full-dimensional, rational inside-out polytope, then

$$E^{\circ}_{\mathcal{P},\mathcal{H}}(k) = \sum_{u \in \mathcal{L}(\mathcal{H})} \mu(\mathbb{R}^d, u)\ \mathrm{Ehr}_{\mathcal{P} \cap u}(k)\,,$$

and if $\mathcal{H}$ is transverse to $\mathcal{P}$

$$E_{\mathcal{P},\mathcal{H}}(k) = \sum_{u \in \mathcal{L}(\mathcal{H})} |\mu(\mathbb{R}^d, u)|\ \mathrm{Ehr}_{\mathcal{P} \cap u}(k)\,.$$

($\mathcal{H}$ is transverse to $\mathcal{P}$ if every flat $u \in \mathcal{L}(\mathcal{H})$ that intersects $\mathcal{P}$ also intersects P°, and $\mathcal{P}$ does not lie in any of the hyperplanes of $\mathcal{H}$.)

Flow Polynomials

A nowhere-zero k-flow on a graph $\Gamma = (V, E)$ is a mapping

$$x : E \to \{-k+1, -k+2, \ldots, -2, -1, 1, 2, \ldots, k-2, k-1\}$$

such that for every node $v \in V$

$$\sum_{h(e)=v} x(e) = \sum_{t(e)=v} x(e)$$

$$\begin{array}{lcl} h(e) & := & \text{head} \\ t(e) & := & \text{tail} \end{array} \quad \text{of the edge } e \text{ in a (fixed) orientation of } \Gamma$$

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Theorem (Kochol 2002)
$\varphi_\Gamma(k) := \#\,(\text{nowhere-zero } k\text{-flows})$ is a polynomial in k.

Flow Polynomial Reciprocity

Let C denote the subspace of $\mathbb{R}^E$ determined by the flow-conservation equations, $\mathcal{P} := [-1, 1]^E \cap C$, and $\mathcal{H}$ the arrangement of all coordinate hyperplanes in $\mathbb{R}^E$. Then $\varphi_\Gamma(k) = E^\circ_{\mathcal{P}^\circ, \mathcal{H}}(k)$.

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Greene–Zaslavsky's Observation Every region of the hyperplane arrangement induced by $\mathcal{H}$ in C corresponds to a totally cyclic orientation.

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(An orientation of Γ is totally cyclic if every edge lies in a coherent circle, that is, where the edges are oriented in a consistent direction around the circle. A totally cyclic orientation τ and a flow x are compatible if $x \geq 0$ when it is expressed in terms of τ.)

Theorem $(-1)^{|E|-|V|+c(\Gamma)}\varphi_\Gamma(-k)$ equals the number of pairs (τ, x) consisting of a totally cyclic orientation τ and a compatible $(k+1)$-flow x. In particular, the constant term $\varphi_\Gamma(0)$ equals the number of totally cyclic orientations of Γ.

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- Prove that every graph admits a 5-flow.

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